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The generalization of bivariate MKZ operators by multiple generating functions

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Abstract

In the present paper, we study approximation properties of multiple generating functions type bivariate Meyer-König and Zeller (MKZ) operators with the help of Volkov type theorem. We compute the order of convergence of these operators by means of modulus of continuity and the elements of modified Lipschitz class. Finally, we give application to partial differential equations.

Keywords: Positive linear operators; Volkov type theorem; Bivariate Meyer-König and Zeller operators; Modified Lipschitz class; Modulus of continuity

1. Introduction

For a function defined on [0, 1) the Meyer-König and Zeller operators [12] and Berstein power series defined by Cheney and Sharma [5] are given by

$$M_n(f;x) = (1-x)^{n+1}\sum_{k=0}^{\infty} f\left(\frac{k}{k+n+1}\right)\binom{n+k}{k}x^k \tag{1.1}$$

and

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$$M_n^*(f;x)=(1-x)^{n+1}\sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right)\binom{n+k}{k}x^k, \tag{1.2}$$

respectively. Recently, some generalizations of these operators were considered in [1,2,6,8,9]. Doğru [6] defined a sequence of generalized linear positive operators which includes MKZ operators and gave some approximation properties of these operators. Agratini [2] defined another sequence of generalized linear positive operators which includes the sequence obtained in [6] and proved that the sequence converges to the identity operator. Gupta [8] introduced a integral modification of the Meyer-König and Zeller Bezier type operators and estimated the rate of convergence of functions of bounded variation. Gupta [9] also introduced Bezier variant of the MKZ operators and studied the rate of convergence by means of the decomposition technique of functions of bounded variation. In [1], Abel, Gupta and Ivan considered a general Durrmeyer variant of the MKZ operators and derived the complete asymptotic expansion for these operators. More recently, Altın, Doğru and Taşdelen [4] introduced a generalization of (1.1) and (1.2) by linear generating functions as follows:

$$L_n(f;x)=\frac{1}{h_n(x,t)}\sum_{k=0}^{\infty} f\left(\frac{a_{k,n}}{a_{k,n}+b_n}\right)C_{k,n}(t)x^k \tag{1.3}$$

for $0\leqslant \frac{a_{k,n}}{a_{k,n}+b_n}\leqslant A$, $A\in(0,1)$, where $\{h_n(x,t)\}_{n\in\mathbb{N}}$ is the generating functions for the sequence of functions $\{C_{k,n}(t)\}_{k\in\mathbb{N}_0}$ in the form $h_n(x,t)=\sum_{k=0}^{\infty}C_{k,n}(t)x^k$ for all $t\in I$ that is any subinterval of $\mathbb{R}$. Authors studied some approximation properties of the operators (1.3).

In this paper, we consider the sequence of linear positive operators of two variables

$$L_{n,m}(f;x,y)=\frac{1}{\Psi_{n,m}(x,y,s,t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} f\left(\frac{a_{k,n}}{a_{k,n}+b_n},\frac{c_{l,m}}{c_{l,m}+d_m}\right)\Gamma_{k,l}^{n,m}(s,t)x^k y^l \tag{1.4}$$

for $0\leqslant \frac{a_{k,n}}{a_{k,n}+b_n}\leqslant A$ and $0\leqslant \frac{c_{l,m}}{c_{l,m}+d_m}\leqslant B$; $A,B\in(0,1)$ where $\{\Psi_{n,m}(x,y,s,t)\}_{n,m\in\mathbb{N}}$ is the multiple generating functions (see [13]) for the sequence of functions $\{\Gamma_{k,l}^{n,m}(s,t)\}_{k,l\in\mathbb{N}_0}$ of the form

$$\Psi_{n,m}(x,y,s,t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Gamma_{k,l}^{n,m}(s,t)x^k y^l$$

with $\Gamma_{k,l}^{n,m}(s,t)\geqslant 0$ for all $(s,t)\in D^2\subset\mathbb{R}^2$.

Assume that the following conditions hold:

(1) $\Psi_{n,m}(x,y,s,t)=(1-x)\Psi_{n+1,m}(x,y,s,t)$,
(2) $a_{k+1,n}\Gamma_{k+1,l}^{n,m}(s,t)=b_n\Gamma_{k,l}^{n+1,m}(s,t)$,
(3) $a_{k+1,n}=a_{k,n+1}+\varphi_n$, $|\varphi_n|\leqslant n_1<\infty$ and $a_{0,n}=0$,
(4) $b_n\to\infty$, $\frac{b_{n+1}}{b_n}\to 1$ and $b_n\neq 0$ for all $n\in\mathbb{N}$,
(5) $\Psi_{n,m}(x,y,s,t)=(1-y)\Psi_{n,m+1}(x,y,s,t)$,
(6) $c_{l+1,m}\Gamma_{k,l+1}^{n,m}(s,t)=d_m\Gamma_{k,l}^{n,m+1}(s,t)$,
(7) $c_{l+1,m}=c_{l,m+1}+\phi_m$, $|\phi_m|\leqslant m_1<\infty$ and $c_{0,m}=0$,
(8) $d_m\to\infty$, $\frac{d_{m+1}}{d_m}\to 1$ and $d_m\neq 0$ for all $m\in\mathbb{N}$.

Remark 1. Some particular cases of the operators (1.4) are defined as follows:

Case 1: If we take
(a) $\Psi_{n,m}(x,y,s,t) = h_n(x,t)h_m(y,s)$,
(b) $\Gamma_{k,l}^{n,m}(s,t) = C_{k,n}(t)C_{l,m}(s)$,
then (1.4) becomes a bivariate extension of (1.3).
If also
(c) $f(x,y) = f_1(x)f_2(y)$,
then we have $L_{n,m}(f;x,y) = L_n(f_1;x)L_m(f_2;y)$.

Case 2: If we chose $\Psi_{n,m}(x,y,s,t) = (1-x)^{-n-1}(1-y)^{-m-1}$, $\Gamma_{k,l}^{n,m}(s,t) = \binom{n+k}{k}\binom{l+m}{l}$, $a_{k,n} = k$, $b_n = n+1$, $c_{l,m} = l$ and $d_m = m+1$, then as a result of case 1 the operators (1.4) turn to be Meyer-König and Zeller operators of two variables.

Case 3: If (c) holds and if we take $\Psi_{n,m}(x,y,s,t)$, $\Gamma_{k,l}^{n,m}(s,t)$, $a_{k,n}$, b_n and $c_{l,m}$ as in case 2 and $d_m = m$, then we get $L_{n,m}(f;x,y) = M_n(f_1;x)M_m^*(f_2;y)$.

2. Approximation properties of $L_{n,m}$

In this section, we shall investigate the approximation properties of $L_{n,m}$ with the help of the test functions

$$f_0(u,v) = 1, \qquad f_1(u,v) = \frac{u}{1-u}, \qquad f_2(u,v) = \frac{v}{1-v},$$
$$f_3(u,v) = \left(\frac{u}{1-u}\right)^2 + \left(\frac{v}{1-v}\right)^2. \tag{2.1}$$

Because for the nodes $u = \frac{a_{k,n}}{a_{k,n}+b_n}$ and $v = \frac{c_{l,m}}{c_{l,m}+d_m}$ the denominators of $\frac{u}{1-u} = \frac{a_{k,n}}{b_n}$ and $\frac{v}{1-v} = \frac{c_{l,m}}{d_m}$ are free of k and l, respectively.

Now we define the space in this work. Let $[0,A]\times[0,B] = I^2$ and let $H_w(I^2)$ be a space of real valued functions $f \in C(I^2)$ satisfying

$$\left|f(u,v) - f(x,y)\right| \leqslant w\left(f, \left|\left(\frac{u}{1-u}, \frac{v}{1-v}\right) - \left(\frac{x}{1-x}, \frac{y}{1-y}\right)\right|\right). \tag{2.2}$$

In (2.2), $|(\frac{u}{1-u}, \frac{v}{1-v}) - (\frac{x}{1-x}, \frac{y}{1-y})| = \sqrt{(\frac{u}{1-u} - \frac{x}{1-x})^2 + (\frac{v}{1-v} - \frac{y}{1-y})^2}$ and w is the modulus of continuity of f denoted by for $\delta > 0$,

$$w(f,\delta) = \sup\left\{\left|f(u,v) - f(x,y)\right|;\ (u,v),(x,y) \in I^2,\ \left|(u,v) - (x,y)\right| < \delta\right\}$$

so that the following conditions are satisfied

(i) $w(f,\delta)$ is non-negative and increasing for δ,
(ii) $\lim_{\delta\to 0} w(f,\delta) = 0$.

It is also well known that for each $(u,v) \in I^2$,

$$\left|f(u,v) - f(x,y)\right| \leqslant w(f,\delta)\left(1 + \frac{|(u,v) - (x,y)|}{\delta}\right). \tag{2.3}$$

Note that the value of $L_{n,m}$ at a point $(x,y) \in I^2$ by $L_{n,m}(f(u,v);x,y)$ or simply $L_{n,m}(f;x,y)$.

An extension of Korovkin Theorem [10] for linear positive operators of two variables was given by Volkov [14]. We now prove a Volkov type theorem using the test functions (2.1) for the linear positive operators of two variables acting from $H_w(I^2)$ to $C(I^2)$ for investigation of the approximation properties of the operators (1.4).

Theorem 1. *Let $A_{n,m}$ be a sequence of linear positive operators acting from $H_w(I^2)$ to $C(I^2)$ and satisfying the four conditions*

$$\lim_{n,m\to\infty}\left\|A_{n,m}(f_i;x,y)-f_i(x,y)\right\|_{C(I^2)}=0,\quad i=0,1,2,3,\tag{2.4}$$

where f_0, f_1, f_2 and f_3 are given by (2.1) *and $\|.\|_{C(I^2)}$ denotes the sup norm on the space $C(I^2)$. Then for all $f\in H_w(I^2)$, we have*

$$\lim_{n,m\to\infty}\left\|A_{n,m}(f;x,y)-f(x,y)\right\|_{C(I^2)}=0.$$

Proof. If $f\in H_w(I^2)$, then using the property (ii) in the definition of $H_w(I^2)$, we can write

$$\left|f(u,v)-f(x,y)\right|<\varepsilon\quad\text{for}\sqrt{\left(\frac{u}{1-u}-\frac{x}{1-x}\right)^2+\left(\frac{v}{1-v}-\frac{y}{1-y}\right)^2}<\delta.$$

Since $(u,v),(x,y)\in I^2$ and $f\in C(I^2)$, there exists a positive constant M such that

$$\left|f(u,v)-f(x,y)\right|<2M.$$

For $\sqrt{(\frac{u}{1-u}-\frac{x}{1-x})^2+(\frac{v}{1-v}-\frac{y}{1-y})^2}\geqslant\delta$ since $(\frac{u}{1-u}-\frac{x}{1-x})^2+(\frac{v}{1-v}-\frac{y}{1-y})^2\geqslant\delta^2$ we can get

$$\left|f(u,v)-f(x,y)\right|<\frac{2M}{\delta^2}\left[\left(\frac{u}{1-u}-\frac{x}{1-x}\right)^2+\left(\frac{v}{1-v}-\frac{y}{1-y}\right)^2\right].$$

Therefore for all $(u,v),(x,y)\in I^2$ and $f\in C(I^2)$, we have

$$\left|f(u,v)-f(x,y)\right|<\varepsilon+\frac{2M}{\delta^2}\left[\left(\frac{u}{1-u}-\frac{x}{1-x}\right)^2+\left(\frac{v}{1-v}-\frac{y}{1-y}\right)^2\right].\tag{2.5}$$

Using linearity and positivity of the operators $A_{n,m}$ and (2.5) one gets

$$\begin{aligned}&\left|A_{n,m}(f;x,y)-f(x,y)\right|\\&\quad\leqslant A_{n,m}\left(\left|f(u,v)-f(x,y)\right|;x,y\right)+\left|f(x,y)\right|\left|A_{n,m}(f_0;x,y)-f_0(x,y)\right|\\&\quad\leqslant\varepsilon+\left(\varepsilon+M+\frac{2M}{\delta^2}\right)\left|A_{n,m}(f_0;x,y)-f_0(x,y)\right|+\frac{4M}{\delta^2}\left|A_{n,m}(f_1;x,y)-f_1(x,y)\right|\\&\qquad+\frac{4M}{\delta^2}\left|A_{n,m}(f_2;x,y)-f_2(x,y)\right|+\frac{2M}{\delta^2}\left|A_{n,m}(f_3;x,y)-f_3(x,y)\right|.\end{aligned}$$

Hence taking supremum over $(x,y)\in I^2$ and using the conditions (2.4) the proof is completed. □

In the light of Theorem 1, we now prove the following main result.

Theorem 2. *Let $L_{n,m}$ be defined by* (1.4). *Then for all $f\in H_w(I^2)$ we have*

$$\lim_{n,m\to\infty}\left\|L_{n,m}(f;x,y)-f(x,y)\right\|_{C(I^2)}=0.$$

Proof. It is enough to prove the conditions of Theorem 1 which are

$$\lim_{n,m\to\infty}\left\|L_{n,m}(f_i;x,y)-f_i(x,y)\right\|_{C(I^2)}=0,\quad i=0,1,2,3,\tag{2.6}$$

where f_0, f_1, f_2 and f_3 are given by (2.1).

It is clear that

$$L_{n,m}(f_0;x,y)=1.\tag{2.7}$$

By using the conditions (1)–(3), we have

$$\begin{aligned}L_{n,m}(f_1;x,y)&=\frac{1}{b_n}\frac{1}{\Psi_{n,m}(x,y,s,t)}\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}a_{k,n}\Gamma_{k,l}^{n,m}(s,t)x^k y^l\\&=\frac{x}{b_n}\frac{1}{\Psi_{n,m}(x,y,s,t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_{k+1,n}\Gamma_{k+1,l}^{n,m}(s,t)x^k y^l\\&=\frac{x}{\Psi_{n,m}(x,y,s,t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Gamma_{k,l}^{n+1,m}(s,t)x^k y^l\\&=\frac{x}{1-x}.\end{aligned}\tag{2.8}$$

In a similar way that of (2.8) by using (5)–(7) and (1)–(8) we obtain

$$L_{n,m}(f_2;x,y)=\frac{y}{1-y}\tag{2.9}$$

and

$$L_{n,m}(f_3;x,y)=\frac{b_{n+1}}{b_n}\left(\frac{x}{1-x}\right)^2+\frac{\varphi_n}{b_n}\frac{x}{1-x}+\frac{d_{m+1}}{d_m}\left(\frac{y}{1-y}\right)^2+\frac{\phi_m}{d_m}\frac{y}{1-y},\tag{2.10}$$

respectively. Thus from (2.7)–(2.10) we reach to (2.6) which is the desired result. □

3. Rates of convergence

In this section, we compute the rates of convergence of $L_{n,m}(f(u,v);x,y)$ to $f(x,y)$ by means of the modulus of continuity and the elements of modified Lipschitz class.

Theorem 3. *If the operator $L_{n,m}$ is defined by* (1.4)*, then for all $f\in H_w(I^2)$ we have*

$$\left\|L_{n,m}(f;x,y)-f(x,y)\right\|_{C(I^2)}\leqslant\left(1+K^{\frac12}\right)w(f,\delta_{nm}),$$

where $K=\max\{\frac{A}{1-A},(\frac{A}{1-A})^2,\frac{B}{1-B},(\frac{B}{1-B})^2\}$ *and* $\delta_{nm}=(\frac{b_{n+1}}{b_n}+\frac{d_{m+1}}{d_m}-2+\frac{\varphi_n}{b_n}+\frac{\phi_m}{d_m})^{\frac12}$.

Proof. By linearity and monotonicity of $L_{n,m}$ and using (2.3) we obtain

$$\begin{aligned}&\left|L_{n,m}(f;x,y)-f(x,y)\right|\\&\quad\leqslant L_{n,m}\left(\left|f(u,v)-f(x,y)\right|;x,y\right)\\&\quad=\frac{1}{\Psi_{n,m}(x,y,s,t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left|f\left(\frac{a_{k,n}}{a_{k,n}+b_n},\frac{c_{l,m}}{c_{l,m}+d_m}\right)-f(x,y)\right|\Gamma_{k,l}^{n,m}(s,t)x^k y^l\end{aligned}$$

$$\leqslant \frac{w(f,\delta_{nm})}{\Psi_{n,m}(x,y,s,t)} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(1+\frac{\sqrt{(\frac{a_{k,n}}{b_n}-\frac{x}{1-x})^2+(\frac{c_{l,m}}{d_m}-\frac{y}{1-y})^2}}{\delta_{nm}}\right)\Gamma_{k,l}^{n,m}(s,t)x^k y^l$$

$$= w(f,\delta_{nm})\Bigg\{1+\frac{1}{\delta_{nm}}\Bigg[\frac{1}{\Psi_{n,m}(x,y,s,t)}$$

$$\times \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sqrt{\left(\frac{a_{k,n}}{b_n}-\frac{x}{1-x}\right)^2+\left(\frac{c_{l,m}}{d_m}-\frac{y}{1-y}\right)^2}\,\Gamma_{k,l}^{n,m}(s,t)x^k y^l\Bigg]\Bigg\}.$$

By the Cauchy–Schwarz inequality and Theorem 2, we get

$$\left|L_{n,m}(f;x,y)-f(x,y)\right|$$

$$\leqslant w(f,\delta_{nm})\Bigg\{1+\frac{1}{\delta_{nm}}\Bigg[\frac{1}{\Psi_{n,m}(x,y,s,t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Bigg(\left(\frac{a_{k,n}}{b_n}-\frac{x}{1-x}\right)^2$$

$$+\left(\frac{c_{l,m}}{d_m}-\frac{y}{1-y}\right)^2\Bigg)\Gamma_{k,l}^{n,m}(s,t)x^k y^l\Bigg]^{\frac{1}{2}}\Bigg\}$$

$$= w(f,\delta_{nm})\Bigg\{1+\frac{1}{\delta_{nm}}\Bigg[\left(\frac{b_{n+1}}{b_n}-1\right)\left(\frac{x}{1-x}\right)^2+\left(\frac{d_{m+1}}{d_m}-1\right)\left(\frac{y}{1-y}\right)^2$$

$$+\frac{\varphi_n}{b_n}\frac{x}{1-x}+\frac{\phi_m}{d_m}\frac{y}{1-y}\Bigg]^{\frac{1}{2}}\Bigg\}.$$

Thus taking supremum over $(x,y)\in I^2$ we obtain the proof of the theorem. □

We will now study the rates of convergence of the positive linear operators $L_{n,m}$ by means of the elements of the modified Lipschitz class $\widetilde{\mathrm{Lip}}_M(\alpha)$ for $0<\alpha\leqslant 1$. As a first step we give the definition of the modified Lipschitz class (see [4]) for the functions of two variables. Consider the class of functions defined by $\widetilde{\mathrm{Lip}}_M(\alpha)$,

$$\left|f(u,v)-f(x,y)\right|\leqslant M\left|\left(\frac{u}{1-u},\frac{v}{1-v}\right)-\left(\frac{x}{1-x},\frac{y}{1-y}\right)\right|^{\alpha} \tag{3.1}$$

for $(u,v),(x,y)\in I^2$ where $M>0$ and $f\in C(I^2)$. We can call the class $\widetilde{\mathrm{Lip}}_M(\alpha)$ as "the modified Lipschitz class." We point out that $\mathrm{Lip}_M(\alpha)\subset\widetilde{\mathrm{Lip}}_M(\alpha)$.

Theorem 4. *Let $L_{n,m}$ be given by* (1.4). *Then for all $f\in\widetilde{\mathrm{Lip}}_M(\alpha)$, we have*

$$\left\|L_{n,m}(f;x,y)-f(x,y)\right\|_{C(I^2)}\leqslant MK^{\frac{\alpha}{2}}\delta_{nm}^{\alpha},$$

where K and δ_{nm} are the same as in Theorem 3.

Proof. Let $f\in\widetilde{\mathrm{Lip}}_M(\alpha)$. By linearity and monotonicity of $L_{n,m}$ and (3.1) we have

$$\left|L_{n,m}(f;x,y)-f(x,y)\right|$$

$$\leqslant \frac{M}{\Psi_{n,m}(x,y,s,t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\left(\frac{a_{k,n}}{b_n}-\frac{x}{1-x}\right)^2+\left(\frac{c_{l,m}}{d_m}-\frac{y}{1-y}\right)^2\right]^{\frac{\alpha}{2}}\Gamma_{k,l}^{n,m}(s,t)x^k y^l.$$

Applying the Hölder inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$, we get

$$\begin{aligned}&\left|L_{n,m}(f; x, y) - f(x, y)\right| \\ &\quad \leqslant M\left[\frac{1}{\Psi_{n,m}(x, y, s, t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\left(\frac{a_{k,n}}{b_n} - \frac{x}{1-x}\right)^2 + \left(\frac{c_{l,m}}{d_m} - \frac{y}{1-y}\right)^2\right)\right. \\ &\qquad \left. \times \Gamma_{k,l}^{n,m}(s, t)x^k y^l\right]^{\frac{\alpha}{2}}.\end{aligned}$$

As in Theorem 3, by using Theorem 2 and taking the supremum over $(x, y) \in I^2$ we arrive at the desired result. □

4. Application to partial differential equations

We consider the operators

$$L^*_{n,m}(f; x, y) = \frac{1}{\Psi_{n,m}(x, y, s, t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} f\left(\frac{k}{k+b_n}, \frac{l}{l+d_m}\right)\Gamma_{k,l}^{n,m}(s, t)x^k y^l \tag{4.1}$$

which is the special case of (1.4) when taking $a_{k,n} = k$ and $c_{l,m} = l$.

In this section by Theorem 5, we shall give a partial differential equation for the bivariate operators (4.1) that seems to be fundamental for the investigation of several kinds of linear positive operators. There are some papers in which differential equations are given for some linear positive operators. For example, see [3,4,6,7,11,15].

Theorem 5. *Let*

$$\frac{\partial}{\partial x}\big(\Psi_{n,m}(x, y, s, t)\big) = K_n(x)\Psi_{n,m}(x, y, s, t), \tag{4.2}$$

$$\frac{\partial}{\partial y}\big(\Psi_{n,m}(x, y, s, t)\big) = H_m(y)\Psi_{n,m}(x, y, s, t) \tag{4.3}$$

and $g(u, v) = \frac{u}{1-u} + \frac{v}{1-v}$. *Then for each* $(x, y) \in I^2$ *and* $f \in H_w(I^2)$, $L^*_{n,m}$ *as defined in* (4.1) *satisfies the functional partial differential equation*

$$\begin{aligned}\left(\frac{x}{b_n}\frac{\partial}{\partial x} + \frac{y}{d_m}\frac{\partial}{\partial y}\right)L^*_{n,m}(f; x, y) &= -\left(\frac{x}{b_n}K_n(x) + \frac{y}{d_m}H_n(y)\right)L^*_{n,m}(f; x, y) \\ &\quad + L^*_{n,m}(fg; x, y).\end{aligned}$$

Proof. Since $f \in C(I^2)$, the power series on the right side of (4.1) converges on I^2. Hence we can differentiate partially it term by term in I^2. So differentiating with respect to x and using (4.2) we have

$$\begin{aligned}\frac{\partial}{\partial x}L^*_{n,m}(f; x, y) &= -K_n(x)L^*_{n,m}(f; x, y) \\ &\quad + \frac{1}{\Psi_{n,m}(x, y, s, t)}\sum_{k=1}^{\infty}\sum_{l=0}^{\infty} kf\left(\frac{k}{k+b_n}, \frac{l}{l+d_m}\right)\Gamma_{k,l}^{n,m}(s, t)x^{k-1}y^l.\end{aligned}$$

Multiplying both sides of this by $\frac{x}{b_n}$ we obtain

$$\frac{x}{b_n}\frac{\partial}{\partial x}L^*_{n,m}(f;x,y) = -\frac{x}{b_n}K_n(x)L^*_{n,m}(f;x,y) + \frac{1}{\Psi_{n,m}(x,y,s,t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{k}{b_n}f\left(\frac{k}{k+b_n},\frac{l}{l+d_m}\right)\Gamma_{k,l}^{n,m}(s,t)x^k y^l. \tag{4.4}$$

In a similar manner differentiating (4.1) with respect to y and using (4.3), we have

$$\frac{y}{d_m}\frac{\partial}{\partial y}L^*_{n,m}(f;x,y) = -\frac{y}{d_m}H_m(y)L^*_{n,m}(f;x,y) + \frac{1}{\Psi_{n,m}(x,y,s,t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{l}{d_m}f\left(\frac{k}{k+b_n},\frac{l}{l+d_m}\right)\Gamma_{k,l}^{n,m}(s,t)x^k y^l. \tag{4.5}$$

Summing (4.4) and (4.5) side by side and using $(\frac{k}{b_n}+\frac{l}{d_m}) = g(\frac{k}{k+b_n},\frac{l}{l+d_m})$, it follows that

$$\begin{aligned}
&\left(\frac{x}{b_n}\frac{\partial}{\partial x}+\frac{y}{d_m}\frac{\partial}{\partial y}\right)L^*_{n,m}(f;x,y)\\
&\quad= -\left(\frac{x}{b_n}K_n(x)+\frac{y}{d_m}H_m(y)\right)L^*_{n,m}(f;x,y)\\
&\qquad+\frac{1}{\Psi_{n,m}(x,y,s,t)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}f\left(\frac{k}{k+b_n},\frac{l}{l+d_m}\right)g\left(\frac{k}{k+b_n},\frac{l}{l+d_m}\right)\Gamma_{k,l}^{n,m}(s,t)x^k y^l.
\end{aligned}$$

In view of (4.1), we obtain the required result. □

Remark 2. Replace I^2 by $I^n = [0,A_1]\times\cdots\times[0,A_n]$ where $A_1,\ldots,A_n \in (0,1)$ and consider the modulus of continuity $w(f,\delta)$ for the functions f of n-variables, given by for $\delta > 0$,

$$\begin{aligned}
w(f,\delta) = \sup\bigl\{&\bigl|f(u_1,\ldots,u_n)-f(x_1,\ldots,x_n)\bigr|;\\
&(u_1,\ldots,u_n),(x_1,\ldots,x_n)\in I^n,\ \bigl|(u_1,\ldots,u_n)-(x_1,\ldots,x_n)\bigr|<\delta\bigr\}.
\end{aligned}$$

Then let $H_w(I^n)$ be space of all real valued functions satisfying

$$\begin{aligned}
&\bigl|f(u_1,\ldots,u_n)-f(x_1,\ldots,x_n)\bigr|\\
&\quad\leqslant w\left(f,\left|\left(\frac{u_1}{1-u_1},\ldots,\frac{u_n}{1-u_n}\right)-\left(\frac{x_1}{1-x_1},\ldots,\frac{x_n}{1-x_n}\right)\right|\right).
\end{aligned}$$

Now consider the class of functions defined by $\widetilde{\mathrm{Lip}}_M(\alpha)$,

$$\bigl|f(u_1,\ldots,u_n)-f(x_1,\ldots,x_n)\bigr| \leqslant M\left|\left(\frac{u_1}{1-u_1},\ldots,\frac{u_n}{1-u_n}\right)-\left(\frac{x_1}{1-x_1},\ldots,\frac{x_n}{1-x_n}\right)\right|^{\alpha}$$

for $(u_1,\ldots,u_n),(x_1,\ldots,x_n)\in I^n$, where $M>0$, $0<\alpha\leqslant 1$ and $f\in C(I^n)$.

Thus all results in this paper can be extended to the case of n-variate functions.

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