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ON APPROXIMATION TO DISCRETE Q-DERIVATIVES OF FUNCTIONS VIA Q-BERNSTEIN-SCHURER OPERATORS

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Abstract. In the present paper, we shall investigate the pointwise approximation properties of the $q-$analogue of the Bernstein-Schurer operators and estimate the rate of pointwise convergence of these operators to the functions f whose $q-$derivatives are bounded variation on the interval $[0,1+p]$. We give an estimate for the rate of convergence of the operator $(B_{n,p,q}f)$ at those points x at which the one sided $q-$derivatives $D_q^+f(x)$ and $D_q^-f(x)$ exist. We shall also prove that the operators $(B_{n,p,q}f)(x)$ converge to the limit $f(x)$. As a continuation of the very recent and initial study of the author deals with the pointwise approximation of the $q-$Bernstein Durrmeyer operators [12] at those points x at which the one sided $q-$derivatives $D_q^+f(x)$ and $D_q^-f(x)$ exist, this study provides (or presents) a forward work on the approximation of q-analogue of the Schurer type operators in the space of D_qBV.

1. Introduction.

Quantum calculus (or q analysis) has attracted the attention of many researchers, because it is revealed to be very suitable for the applications in various fields, such as quantum mechanics, mathematical physics, nuclear and high energy physics with the aim of understanding black holes, functional analysis and especially in the last decades approximation theory and so on (see [2]-[3], [11]-[14], [19] and the references therein).

Due to the versatility and importance of the theory of quantum calculus, it has been studied thanks to the crucial contribution of the so-called $q-$analogue of the approximation operators in Functional Analysis. As a pioneering work, in 1997 Phillips [18] introduced the generalization of Bernstein polynomials based on $q-$integers to prove the approximation (or superposition) problem of Weierstrass. It is useful to point out that, the $q-$analogue of the Bernstein operators was defined by Lupas [15].

In view of the fundamental studies due to Phillips [18], [19] and Lupas [15], in 2011 Muraru [17] introduced the generalization of $q-$Bernstein-Schurer operators as follows:

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Let $p \in \mathbb{N}_0$ be fixed. For a function defined on the interval $[0, 1+p]$, the generalized $q-$Bernstein-Schurer operators $B_{n,p,q}(f)$ are defined by

$$(B_{n,p,q}f)(x) = \sum_{k=0}^{n+p} f\left(\frac{[k]_q}{[n]_q}\right) p_{k,n,p,q}(x), 0 \leq x \leq 1 \quad , \quad n \geq 1, \tag{1}$$

where

$$p_{k,n,p,q}(x) = \left[\begin{array}{c} n+p \\ k \end{array}\right]_q x^k \prod_{s=0}^{n+p-k-1} (1-q^s x), 0 \leq x \leq 1$$

or equivalently

$$p_{k,n,p,q}(x) = \left[\begin{array}{c} n+p \\ k \end{array}\right]_q x^k (1-x)_q^{n+p-k}.$$

The operators (1) and their different modifications were studied by many researchers (see e.g. [2], [20] and [1] etc.).

It can be easily verified that in case $q = 1$ and $p = 0$, the operators defined by (1) reduce to the celebrated Bernstein operators.

It is well-known that, especially last two decades, several authors investigated the convergence problems for linear positive operators in the spaces of $BV(I)$ and $DBV(I)$, where $I \subset \mathbb{R}$ (See [5]-[7]).

To the best of the author's knowledge, until a very recent study [12], no mathematician investigated the convergence of the $q-$operators in D_qBV spaces using the $q-$left and $q-$right derivatives. In [12], Karsli estimated the rate of convergence of $q-$Bernstein-Durrmeyer Operators for functions in $D_qBV[0,\infty)$. As a continuation of this initial study, the main purpose of this paper is to study the pointwise behavior of the $q-$Bernstein-Schurer operators on $[0, 1+p]$. In particular, we give an estimate for the rate of convergence of the operator $(B_{n,p,q}f)$ at those points x at which the one sided $q-$derivatives $D_q^+ f(x)$ and $D_q^- f(x)$ exist. We prove that the operators $B_{n,p,q}f$ converges to the limit $f(x)$ in the space of D_qBV.

The class of real valued $q-$differentiable functions defined on a set $[a, b]$, whose $q-$derivatives are bounded variation on $[a, b]$, is denoted by $D_qBV[a, b]$. The elements of this space can be written as

$$f(x) = C + \int_a^x \Psi(t) d_q t, \; -\infty < a \leq x \leq b$$

where C is a constant and $\Psi \in BV[a, b]$. It is clear that

$$D_qBV[a, b] := \{f : D_q f = \Psi \in BV[a, b]\}.$$

Here, we establish an estimate for the aforementioned $q-$Bernstein-Schurer operators (1). The first theorem of this paper is stated as:

Theorem 1. *Let f be a bounded function on $[0, 1+p]$. Suppose that the right and left $q-$derivatives exist at a fixed point $x \in (0, 1)$. Then at this point $x \in (0, 1)$, and n sufficiently large, one has*

$$\left|(B_{n,p,q}f)(x) - f(x)\right| \leq \left|\frac{D_q^+ f(x) - D_q^- f(x)}{2}\right| \sqrt{A_{n,p,q}(x)} \tag{2}$$

$$+(1-q)\sum_{l=0}^{\infty} q^l \left[\frac{1}{\sqrt{[n]_q}} \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}} (D_q f_x) + \frac{1}{[n]_q} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x-\frac{x}{k}}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{k}} (D_q f_x) \right]$$

where

$$D_q f_x(t) = \begin{cases} D_q f(t) - D_q^+ f(x) & , \quad x < t \leq 1+p \\ 0 & , \quad t = x \\ D_q f(t) - D_q^- f(x) & , \quad 0 \leq t < x \end{cases}, \tag{3}$$

$\bigvee_a^b (f)$ *is the total variation (or Jordan Variation) of* f *on* $[a,b]$ *and*

$$A_{n,p,q}(x) := \left(B_{n,p,q}(t-x)^2\right)(x).$$

2. **Auxiliary results.** In this section, we state some basic concepts concerning quantum calculus and some lemmas about aforementioned operators, which are necessary to prove our theorems.

Now, we give a brief list of some well-known and important definitions, such as $q-$number, $q-$Binomial coefficient, $q-$derivative and $q-$integral, together with their properties of $q-$calculus, which are required in this paper.

For any fixed real number $q > 0$ and non-negative integer r, the $q-$integer of the number r is defined by

$$[r]_q = \begin{cases} (1-q^r)/(1-q) & , \quad q \neq 1 \\ r & , \quad q = 1 \end{cases}.$$

The $q-$factorial is defined by

$$[r]_q! = \begin{cases} [r]_q [r-1]_q .....[1]_q & , \quad r = 1,2,3,..... \\ 1 & , \quad r = 0. \end{cases},$$

and $q-$Binomial coefficient is defined as

$$\left[\begin{array}{c} n \\ r \end{array} \right]_q = \frac{[n]_q!}{[r]_q![n-r]_q!},$$

for integers $n \geq r \geq 0$.

Let $q \in (0,1)$ and I be a real interval containing 0.

Definition 1. Let $f : I \to R$ be a function and let $x \in I$. The Jackson's $q-$derivative D_q (see [11], [3]), or Jackson's difference operator, of a function f at x is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \;\; if \;\; x \neq 0, \qquad D_q f(0) := \lim_{x \to 0} D_q f(x). \tag{4}$$

In addition, we note that

$$D_q f(x) \to f'(x)$$

as q tends to 1, provided $f'(x)$ exists.

Definition 2. Let $f : I \to R$ be a function with $x \in I$. The left (backward), right (forward) and symmetric $q-$derivatives of a function f are given by, respectively,

$$D_q^- f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \tag{5}$$

$$D_q^+ f(x) := \frac{f\left(\frac{x}{q}\right) - f(x)}{(1-q)x} \tag{6}$$

and

$$D_q^s f(x) := \frac{f(qx) - f\left(\frac{x}{q}\right)}{\left(q - \frac{1}{q}\right)x}. \tag{7}$$

provided that $x \neq 0$. (See [14], [12]).

Note that $q-$calculus is an interdisciplinary subject. As a result many different definitions of the above derivatives have been used. For example, the left (backward) $q-$derivative of f at $x \neq 0$ is defined by the quotient

$$D_q^- f(x) := \frac{f(x) - f\left(\frac{x}{q}\right)}{\left(1 - \frac{1}{q}\right)x}. \tag{8}$$

(See [14], [12]).

Remark 1. The Jackson's $q-$derivative is sometimes called the left (backward) $q-$derivative. Namely

$$D_q f(x) = D_q^- f(x).$$

It is clear that if a function f is differentiable (in the classical sense) at x, then

$$\lim_{q \to 1^-} D_q^- f(x) = \lim_{q \to 1^-} D_q^+ f(x) = f'(x),$$

holds true, where the symbol f' denotes the usual derivative.

The generalized form of the backward and forward $q-$derivatives and their relations can be found in [16].

Remark 2. We note that, a continuous function on an interval, which does not include zero, is continuous $q-$differentiable.

Remark 3. Although the difference operators convey the same idea, it turns out that their proper choice in constructing the Fourier transform between configuration and momentum space [[8], p. 1797] and solving $q-$diffusion equations by Laplace and Mellin transform [[9], sections 2-3].

Example with informations. Let us consider a $q-$diffusion equation as;

$$D_t^q y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t), \qquad (-\infty < x < \infty, \quad t > 0)$$

with the initial condition

$$y(x,0) = f(x).$$

The main question now is how to choose an appropriate integral transform to remove the $q-$derivative, namely how can we solve the aforementioned $q-$diffusion equation. In view of the positivity of the time variable, two natural choices are the Laplace and the Mellin transforms.

If one use the Laplace transform of the $q-$derivatives given in (5) and (8), and trying to solve this $q-$diffusion equation, then Ho [[9], sections 2] informed us that the Laplace transform is not useful in solving equations involving $q-$derivatives.

Alternatively, if we use the Mellin transform of the $q-$derivatives given in (5) and (8), and trying to solve the $q-$diffusion equation, then Ho [[9], sections 3] showed that the $q-$derivative given in (8) more useful and simpler than the use $q-$derivative given in (5).

From the definition of the $q-$derivative one can obtain the $q-$derivative of a product as follows;

$$D_q\left(f(x)g(x)\right)=D_q\left(f(x)\right)g(x)+f(qx)D_q\left(g(x)\right).$$

Now, we mention about $q-$integral or the so called Jackson $q-$integral.

The $q-$analogue of integration, independently discovered by Thomae [21] and Jackson [10], in the interval $[0,a]$ is defined by

$$\int_0^a f(t)d_qt := a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n \qquad 0<q<1, \tag{9}$$

and over a general interval:

$$\int_c^d f(t)d_qt := \int_0^d f(t)d_qt - \int_0^c f(t)d_qt.$$

Some properties of the $q-$integration are given as:

1- From $q-$derivative of a product, one can also obtain the following formula of $q-$integration by parts

$$\int_c^d f(t)D_qg(t)d_qt = f(t)g(t)\mid_c^d - \int_c^d g(qt)D_qf(t)d_qt$$

provided f and g are continuous on $[c,d]$.

2- For continuous functions $f(t)$ one can easily see that $q-$integral approaches the Riemann integral as $q\to 1$, and also that the operators of $q-$differentiation and indefinite $q-$integration are inverse to each other, namely

$$D_{q,t}\int f(t)d_qt = f(t) = \int D_{q,t}f(t)d_qt.$$

3- If F is any anti $q-$derivative of the function f, namely $D_qF=f$, continuous at $t=0$, then

$$\int_0^a f(t)d_qt = F(a)-F(0).$$

Lemma 1 [17]. *Using the definition of $q-$Bernstein-Schurer operators, we obtain*

$$\left(B_{n,p,q}1\right)(x)=1,\quad \left(B_{n,p,q}t\right)(x)=\frac{x[n+p]_q}{[n]_q}$$

and

$$\left(B_{n,p,q}t^2\right)(x)=\left(\frac{[n+p]_q}{[n]_q}\right)^2x^2+\frac{X[n+p]_q}{[n]_q^2},$$

where $X=x(1-x)$.

Remark 4. We observe that,

$$\left(B_{n,p,q}(t-x)\right)(x)=\frac{x[n+p]_q}{[n]_q}-x=x\left(\frac{[n+p]_q}{[n]_q}-1\right), \tag{10}$$

$$\begin{aligned}\left(B_{n,p,q}(t-x)^2\right)(x) &= x^2\left[\left(\frac{[n+p]_q}{[n]_q}\right)^2 - 2\frac{[n+p]_q}{[n]_q}+1\right] \\ +\frac{X[n+p]_q}{[n]_q^2} &: = A_{n,p,q}(x).\end{aligned}$$

Lemma 2. *For all $x \in (0,1)$, and n sufficiently large, then for $0 < t < x$, one obtains*

$$\lambda_{n,p,q}(x,t) = \int_0^t \left(D_{q,t}K_{n,p,q}(x,u)\right) d_qu \leq \frac{A_{n,p,q}(x)}{(x-t)^2}, \tag{11}$$

where

$$K_{n,p,q}(x,u) = \begin{cases} \sum\limits_{k\leq(n+p)u} p_{k,n,p,q}(x) & , 0 < u \leq 1 \\ 0 & , \ u = 0 \end{cases}.$$

Proof. Clearly

$$\begin{aligned}\lambda_{n,p,q}(x,t) &= \int_0^t \left(D_{q,t}K_{n,p,q}(x,u)\right) d_qu \\ &\leq \int_0^t \left(D_{q,t}K_{n,p,q}(x,u)\right)\left(\frac{x-u}{x-t}\right)^2 d_qu \\ &= \frac{1}{(x-t)^2}\int_0^t \left(D_{q,t}K_{n,p,q}(x,u)\right)(x-u)^2 d_qu \\ &= \frac{\left(B_{n,p,q}(u-x)^2\right)(x)}{(x-t)^2}.\end{aligned}$$

By (10), one can easily obtain

$$\lambda_{n,p,q}(x,t) \leq \frac{1}{(x-t)^2}A_{n,p,q}(x).$$

□

Remark 5. By Cauchy-Schwarz-Bunyakowsky inequality, one has

$$\left(B_{n,p,q}\left|t-x\right|\right)(x) \leq \left(\left(B_{n,p,q}(t-x)^2\right)(x)\right)^{\frac{1}{2}} \leq \sqrt{A_{n,p,q}(x)}. \tag{12}$$

3. **Main results.** *Proof of Theorem 1.* We write the difference between $\left(B_{n,p,q}f\right)(x)$ and $f(x)$ as follows;

$$\begin{aligned}\left(B_{n,p,q}f\right)(x) - f(x) &= \sum_{k=0}^{n+p} p_{k,n,p,q}(x) f\left(\frac{[k]_q}{[n]_q}\right) - f(x) \\ &= \int_0^{\frac{[n+p]_q}{[n]_q}} \left[f(t)-f(x)\right]\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt,\end{aligned}$$

where $K_{n,p,q}(x,t)$ being defined in Lemma 2. Note that $D_q f = \Psi \in BV[0, 1+p]$, and hence

$$\begin{aligned}
\left(B_{n,p,q}f\right)(x) - f(x) &= \int_0^x \left[f(t) - f(x)\right]\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \\
&\quad + \int_x^{\frac{[n+p]_q}{[n]_q}} \left[f(t) - f(x)\right]\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \\
&= -\int_0^x \left[\int_t^x D_qf(u)\, d_qu\right]\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \\
&\quad + \int_x^{\frac{[n+p]_q}{[n]_q}} \left[\int_x^t D_qf(u)\, d_qu\right]\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \\
&= -E_{1,p,q}(x) + E_{2,p,q}(x),
\end{aligned}$$

here

$$E_{1,p,q}(x) := \int_0^x \left[\int_t^x D_qf(u)\, d_qu\right]\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \tag{13}$$

and

$$E_{2,p,q}(x) := \int_x^{\frac{[n+p]_q}{[n]_q}} \left[\int_x^t D_qf(u)\, d_qu\right]\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt. \tag{14}$$

In view of the definitions (3)-(8), for any $D_q f = \Psi \in BV[0, 1+p]$, we decompose $D_q f(t)$ as

$$\begin{aligned}
D_qf(t) &= \frac{D_q^+ f(x) + D_q^- f(x)}{2} + \frac{D_q^+ f(x) - D_q^- f(x)}{2} sgn(t-x) \\
&\quad + D_q f_x(t) + \delta_x(t)\left[D_qf(x) - \frac{D_q^+ f(x) + D_q^- f(x)}{2}\right]
\end{aligned} \tag{15}$$

where $\delta_x(t)$ is the Dirac Delta function. If we use (15) in (13) and (14), then the following expressions hold true.

$$\begin{aligned}
&E_{1,p,q}(x) \\
&= \int_0^x \left\{\int_t^x \frac{D_q^+ f(x) + D_q^- f(x)}{2} + D_qf_x(u) + \frac{D_q^+ f(x) - D_q^- f(x)}{2} sgn(u-x)\right. \\
&\quad \left. + \delta_x(u)\left[D_qf(x) - \frac{D_q^+ f(x) + D_q^- f(x)}{2}\right] d_qu\right\}\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt
\end{aligned}$$

and

$$
\begin{aligned}
& E_{2,p,q}(x) \\
= & \int\limits_{x}^{\frac{[n+p]_q}{[n]_q}} \left\{ \int\limits_{x}^{t} \frac{D_q^{+}f(x)+D_q^{-}f(x)}{2} + D_q f_x(u) + \frac{D_q^{+}f(x)-D_q^{-}f(x)}{2} sgn(u-x) \right. \\
& \left. +\delta_x(u) \left[D_q f(x) - \frac{D_q^{+}f(x)+D_q^{-}f(x)}{2} \right] \, d_q u \right\} \left(D_{q,t} K_{n,p,q}(x,t) \right) \, d_q t.
\end{aligned}
$$

At first, we consider $E_{1,p,q}(x)$:

$$
\begin{aligned}
& E_{1,p,q}(x) \\
= & \frac{D_q^{+}f(x)+D_q^{-}f(x)}{2} \int\limits_{0}^{x} (x-t) \left(D_{q,t} K_{n,p,q}(x,t) \right) \, d_q t \\
& + \int\limits_{0}^{x} \left[\int\limits_{x}^{t} D_q f_x(u) \, d_q u \right] \left(D_{q,t} K_{n,p,q}(x,t) \right) \, d_q t \\
& - \frac{D_q^{+}f(x)-D_q^{-}f(x)}{2} \int\limits_{0}^{x} (x-t) \left(D_{q,t} K_{n,q}(x,t) \right) d_q t \\
& + \left[D_q f(x) - \frac{D_q^{+}f(x)+D_q^{-}f(x)}{2} \right] \int\limits_{0}^{x} \left[\int\limits_{x}^{t} \delta_x(u) \, d_q u \right] \left(D_{q,t} K_{n,p,q}(x,t) \right) \, d_q t
\end{aligned}
$$

Since $\int\limits_{x}^{t} \delta_x(u) \, d_q u = 0$, one has

$$
\begin{aligned}
E_{1,p,q}(x) \quad = \quad & \frac{D_q^{+}f(x)+D_q^{-}f(x)}{2} \int\limits_{0}^{x} (x-t) \left(D_{q,t} K_{n,p,q}(x,t) \right) d_q t \qquad (16) \\
& + \int\limits_{0}^{x} \left[\int\limits_{x}^{t} D_q f_x(u) \, d_q u \right] \left(D_{q,t} K_{n,p,q}(x,t) \right) d_q t \\
& - \frac{D_q^{+}f(x)-D_q^{-}f(x)}{2} \int\limits_{0}^{x} (x-t) \left(D_{q,t} K_{n,p,q}(x,t) \right) d_q t.
\end{aligned}
$$

Passing through the similar method, we find

$$\begin{aligned} E_{2,p,q}(x) &= \frac{D_q^+ f(x) + D_q^- f(x)}{2} \int\limits_x^{\frac{[n+p]_q}{[n]_q}} (t-x)\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \qquad (17) \\ &+ \int\limits_x^{\frac{[n+p]_q}{[n]_q}} \left[\int\limits_x^t D_q f_x(u)\, d_qu\right] \left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \\ &- \frac{D_q^+ f(x) - D_q^- f(x)}{2} \int\limits_x^{\frac{[n+p]_q}{[n]_q}} (t-x)\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt. \end{aligned}$$

Collecting (16) and (17) yields,

$$\begin{aligned} &-E_{1,p,q}(x) + E_{2,p,q}(x) \\ =\ & \frac{D_q^+ f(x) + D_q^- f(x)}{2} \int\limits_0^{\frac{[n+p]_q}{[n]_q}} (t-x)\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \\ &+ \frac{D_q^+ f(x) - D_q^- f(x)}{2} \int\limits_0^{\frac{[n+p]_q}{[n]_q}} |t-x|\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \\ &- \int\limits_0^x \left[\int\limits_t^x D_q f_x(u)\, d_qu\right] \left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \qquad (18) \\ &+ \int\limits_x^{\frac{[n+p]_q}{[n]_q}} \left[\int\limits_x^t D_q f_x(u)\, d_qu\right] \left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt. \end{aligned}$$

From (18), we can re-write the difference between $(B_{n,p,q}f)(x)$ and $f(x)$,

$$\begin{aligned} &(B_{n,p,q}f)(x) - f(x) \\ =\ & \frac{D_q^+ f(x) + D_q^- f(x)}{2} \int\limits_0^{\frac{[n+p]_q}{[n]_q}} (t-x)\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \\ &+ \frac{D_q^+ f(x) - D_q^- f(x)}{2} \int\limits_0^{\frac{[n+p]_q}{[n]_q}} |t-x|\left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \\ &- \int\limits_0^x \left[\int\limits_t^x D_q f_x(u)\, d_qu\right] \left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt \qquad (19) \\ &+ \int\limits_x^{\frac{[n+p]_q}{[n]_q}} \left[\int\limits_x^t D_q f_x(u)\, d_qu\right] \left(D_{q,t}K_{n,p,q}(x,t)\right) d_qt. \end{aligned}$$

On the other hand

$$\int_{0}^{\frac{[n+p]_q}{[n]_q}} |t-x| \left(D_{q,t}K_{n,p,q}(x,t)\right) \, d_q t = \left(B_{n,p,q} \, |t-x|\right)(x) \tag{20}$$

and

$$\int_{0}^{\frac{[n+p]_q}{[n]_q}} (t-x) \left(D_{q,t}K_{n,p,q}(x,t)\right) \, d_q t = \left(B_{n,p,q} \, (t-x)\right)(x) = 0, \tag{21}$$

are valid, then using (20) and (21) in (19), we get

$$\begin{aligned} \left|\left(B_{n,p,q}f\right)(x) - f(x)\right| \leq & \left|\frac{D_q^+ f(x) + D_q^- f(x)}{2}\right| \left|\left(B_{n,p,q} \, (t-x)\right)(x)\right| \\ & + \left|\frac{D_q^+ f(x) - D_q^- f(x)}{2}\right| \left|\left(B_{n,p,q} \, |t-x|\right)(x)\right| \\ & + \left| - \int_0^x \left[\int_t^x D_q f_x(u) \, d_q u\right] \left(D_{q,t}K_{n,p,q}(x,t)\right) \, d_q t\right| \\ & + \left| \int_x^{\frac{[n+p]_q}{[n]_q}} \left[\int_x^t D_q f_x(u) \, d_q u\right] \left(D_{q,t}K_{n,p,q}(x,t)\right) d_q t.\right| \end{aligned} \tag{22}$$

From the definition of $\lambda_{n,p,q}(x,t)$, we write

$$\begin{aligned} & \int_0^x \left[\int_t^x D_q f_x(u) \, d_q u\right] \left(D_{q,t}K_{n,p,q}(x,t)\right) \, d_q t \\ = & \int_0^x \left[\int_t^x D_q f_x(u) \, d_q u\right] D_{q,t}\lambda_{n,p,q}(x,t) \, d_q t. \end{aligned} \tag{23}$$

Applying partial $q-$integration to the right hand side of (23), one obtains

$$\int_0^x \left[\int_t^x D_q f_x(u) \, d_q u\right] D_{q,t}\lambda_{n,p,q}(x,t) \, d_q t = \int_0^x D_q f_x(t) \, \lambda_{n,p,q}(x,t) \, d_q t.$$

Thus

$$\left| - \int_0^x \left[\int_x^t D_q f_x(u) \, d_q u\right] \left(d_{q,t}K_{n,p,q}(x,t)\right) \, d_q t\right| \leq \int_0^x \left|D_q f_x(t)\right| \, \lambda_{n,p,q}(x,t) \, d_q t$$

and

$$\left| -\int_0^x \left[\int_x^t D_q f_x(u)\, d_q u \right] (D_{q,t} K_{n,p,q}(x,t))\, d_q t \right|$$
$$\leq \int_0^{x-\frac{x}{\sqrt{n}}} |D_q f_x(t)|\ \lambda_{n,p,q}(x,t)\, d_q t + \int_{x-\frac{x}{\sqrt{n}}}^x |D_q f_x(t)|\ \lambda_{n,p,q}(x,t)\, d_q t.$$

Since $D_q f_x(x) = 0$ and $\lambda_{n,p,q}(x,t) \leq 1$,

$$\begin{aligned} & \int_{x-\frac{x}{\sqrt{[n]_q}}}^x |D_q f_x(t)|\ \lambda_{n,p,q}(x,t)\, d_q t \\ = & \int_{x-\frac{x}{\sqrt{[n]_q}}}^x |D_q f_x(t) - D_q f_x(x)|\ \lambda_{n,p,q}(x,t)\, d_q t \\ \leq & \int_{x-\frac{x}{\sqrt{[n]_q}}}^x \bigvee_t^x (D_q f_x)\, d_q t. \end{aligned}$$

Owing to (11), we get

$$\begin{aligned} & \int_0^{x-\frac{x}{\sqrt{[n]_q}}} |D_q f_x(t)|\ \lambda_{n,p,q}(x,t)\, d_q t \\ \leq & \ A_{n,p,q}(x) \int_0^{x-\frac{x}{\sqrt{[n]_q}}} |D_q f_x(t)| \frac{d_q t}{(x-t)^2} \\ \leq & \ A_{n,p,q}(x) \int_0^{x-\frac{x}{\sqrt{[n]_q}}} \bigvee_t^x (D_q f_x) \frac{d_q t}{(x-t)^2}. \end{aligned}$$

Make the change of variables $t = x - \frac{x}{u}$, then one has from (9)

$$\int_{x-\frac{x}{\sqrt{[n]_q}}}^x \bigvee_t^x (D_q f_x)\, d_q t \leq \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^x (D_q f_x) \int_{x-\frac{x}{\sqrt{[n]_q}}}^x d_q t$$

$$= \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^x (D_q f_x) \left[x(1-q) - \left(x - \frac{x}{\sqrt{[n]_q}} \right)(1-q) \right] \sum_{l=0}^{\infty} q^l$$

$$= \frac{x}{\sqrt{[n]_q}} (1-q) \sum_{l=0}^{\infty} q^l \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^x (D_q f_x)$$

and

$$
\begin{aligned}
& A_{n,p,q}(x) \int_{0}^{x-\frac{x}{\sqrt{[n]_q}}} \bigvee_{t}^{x} (D_q f_x) \frac{d_q t}{(x-t)^2} \\
= & \; A_{n,p,q}(x) \int_{1}^{\sqrt{[n]_q}} \bigvee_{x-\frac{x}{u}}^{x} (D_q f_x) \frac{\left(\frac{x}{u^2}\right) d_q u}{(-\frac{x}{u})^2} \\
= & \; \frac{A_{n,p,q}(x)}{x} \int_{1}^{\sqrt{[n]_q}} \bigvee_{x-\frac{x}{u}}^{x} (D_q f_x) \, d_q u \\
\leq & \; \frac{A_{n,p,q}(x)}{x} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x-\frac{x}{k}}^{x} (D_q f_x) \, (1-q) \sum_{l=0}^{\infty} q^l.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
& \left| -\int_{0}^{x} \left[\int_{t}^{x} D_q f_x(u) \, du \right] (D_{q,t} K_{n,p,q}(x,t)) \, d_q t \right| \\
\leq & \; \frac{x}{\sqrt{[n]_q}} (1-q) \sum_{l=0}^{\infty} q^l \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^{x} (D_q f_x) \\
& + A_{n,p,q}(x) \frac{1}{x} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x-\frac{x}{k}}^{x} (D_q f_x) \, (1-q) \sum_{l=0}^{\infty} q^l. \qquad (24)
\end{aligned}
$$

By the same way,

$$
\begin{aligned}
& \int_{x}^{\frac{[n+p]_q}{[n]_q}} \left[\int_{x}^{t} D_q f_x(u) \, d_q u \right] (D_{q,t} K_{n,p,q}(x,t)) \, d_q t \\
\leq & \int_{x}^{\frac{[n+p]_q}{[n]_q}} \left| D_q f_x(t) \right| \, D_{q,t} \lambda_{n,p,q}(x,t) \, d_q t \\
\leq & \int_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}} \left| D_q f_x(t) \right| \, dt + A_{n,p,q}(x) \int_{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}}^{\frac{[n+p]_q}{[n]_q}} \left| D_q f_x(t) \right| \frac{d_q t}{(x-t)^2}
\end{aligned}
$$

$$= \int_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}} |D_q f_x(t) - D_q f_x(x)| \, d_q t$$

$$+A_{n,p,q}(x) \int_{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}}^{\frac{[n+p]_q}{[n]_q}} |D_q f_x(t) - D_q f_x(x)| \frac{d_q t}{(x-t)^2}$$

$$\leq \bigvee_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}} (1-q) \sum_{l=0}^{\infty} q^l$$

$$+A_{n,p,q}(x) \int_{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}}^{\frac{[n+p]_q}{[n]_q}} \bigvee_{x}^{t} (D_q f_x) \frac{d_q t}{(x-t)^2}$$

Make the change of variables $t = x + \frac{\frac{[n+p]_q}{[n]_q}-x}{u}$, again from (9)

$$\leq \bigvee_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}} (1-q) \sum_{l=0}^{\infty} q^l$$

$$+A_{n,p,q}(x) \int_{\sqrt{[n]_q}}^{1} \bigvee_{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{u}}^{x} (D_q f_x) \frac{-\left(\frac{\frac{[n+p]_q}{[n]_q}-x}{u^2}\right) d_q u}{\left(\frac{x-\frac{[n+p]_q}{[n]_q}}{u}\right)^2}$$

$$= \bigvee_{x}^{x+\frac{1-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}} (1-q) \sum_{l=0}^{\infty} q^l$$

$$+\frac{A_{n,p,q}(x)}{\left(\frac{[n+p]_q}{[n]_q}-x\right)} \int_{1}^{\sqrt{[n]_q}} \bigvee_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{u}} (D_q f_x) \, d_q u$$

$$\leq \bigvee_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}} (1-q) \sum_{l=0}^{\infty} q^l$$

$$+ \frac{A_{n,p,q}(x)}{\left(\frac{[n+p]_q}{[n]_q}-x\right)} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \int_{k}^{k+1} \bigvee_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{u}} (D_q f_x)\, d_q u$$

$$\leq \bigvee_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}} (1-q) \sum_{l=0}^{\infty} q^l$$

$$+ \frac{A_{n,p,q}(x)}{\left(\frac{[n+p]_q}{[n]_q}-x\right)} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{k}} (D_q f_x)\,(1-q) \sum_{l=0}^{\infty} q^l.$$

Finally

$$\int_{x}^{\frac{[n+p]_q}{[n]_q}} \left[\int_{x}^{t} D_q f_x(u)\, d_q u\right] (D_{q,t} K_{n,p,q}(x,t))\, d_q t$$

$$\leq \bigvee_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}} (1-q) \sum_{l=0}^{\infty} q^l$$

$$+ \frac{A_{n,p,q}(x)}{\left(\frac{[n+p]_q}{[n]_q}-x\right)} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{k}} (D_q f_x)\,(1-q) \sum_{l=0}^{\infty} q^l. \tag{25}$$

Combining (12), (24) and (25) in (22), we get the desired result (2).

Thus the proof is completed. □

In order to obtain an approximation theorem, we introduce a new sequence with the following properties. Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and

$$\lim_{n\to\infty} q_n = 1.$$

If we replace q by q_n, we have immediately

$$[n]_{q_n} \to \infty \qquad (n \to \infty).$$

It is worthy to mention that similar and detailed approachs can be found [2], [3] and [13].

As a consequence of Theorem 1 and Remark 2, we get:

Corollary 1. *Let $f \in C\,[0,1+p]$. Then for every $x \in (0,1)$, and n sufficiently large, we have*

$$\begin{aligned} &\left|\left(B_{n,p,q}f\right)(x)-f(x)\right| \\ \leq\ & (1-q)\sum_{l=0}^{\infty} q^l \left[\frac{1}{\sqrt{[n]_q}} \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_q}}} (D_q f_x) + \frac{1}{[n]_q}\sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x-\frac{x}{k}}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{k}} (D_q f_x)\right] \end{aligned}$$

On convergence formally, Theorem 2 reads:

Theorem 2. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $\lim_{n\to\infty} q_n = 1$. Let f be a bounded function on $[0,1+p]$. Suppose that the right and left $q-$derivatives exist at a fixed point $x \in (0,1)$. Then at this point $x \in (0,1)$, and n sufficiently large, we have*

$$\left|\left(B_{n,p,q_n}f\right)(x)-f(x)\right| \leq \left|\frac{D_{q_n}^{+}f(x)-D_{q_n}^{-}f(x)}{2}\right|\sqrt{A_{n,p,q_n}(x)}$$

$$+(1-q_n)\sum_{l=0}^{\infty} q_n^l \left[\frac{1}{\sqrt{[n]_{q_n}}} \bigvee_{x-\frac{x}{\sqrt{[n]_{q_n}}}}^{x+\frac{\frac{[n+p]_{q_n}}{[n]_{q_n}}-x}{\sqrt{[n]_{q_n}}}} (D_{q_n} f_x) + \frac{1}{[n]_{q_n}}\sum_{k=1}^{\left[\sqrt{[n]_{q_n}}\right]} \bigvee_{x-\frac{x}{k}}^{x+\frac{\frac{[n+p]_{q_n}}{[n]_{q_n}}-x}{k}} (D_{q_n} f_x)\right].$$

Here we note that

$$A_{n,p,q_n}(x) \to 0 \qquad (n \to \infty).$$

In view of Theorem 2 and Remark 2, we get the following estimate:

Corollary 2. *Let $f \in C[0,1+p]$. Then for every $x \in (0,1)$, and n sufficiently large, we have*

$$\left|\left(B_{n,p,q_n}f\right)(x)-f(x)\right|$$

$$\leq\ (1-q_n)\sum_{l=0}^{\infty} q_n^l \left[\frac{1}{\sqrt{[n]_{q_n}}} \bigvee_{x-\frac{x}{\sqrt{[n]_{q_n}}}}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{\sqrt{[n]_{q_n}}}} (D_{q_n} f_x) + \frac{1}{[n]_{q_n}}\sum_{k=1}^{\left[\sqrt{[n]_{q_n}}\right]} \bigvee_{x-\frac{x}{k}}^{x+\frac{\frac{[n+p]_q}{[n]_q}-x}{k}} (D_{q_n} f_x)\right].$$

The proof of Theorem 2 is the same as Theorem 1, so we omit it.

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