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Fuzzy sets as texture spaces, I. Representation theorems

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Abstract

The authors continue the development of a theory of texture spaces, introducing complemented products and sums, and applying these in a series of representation theorems for fuzzy lattices and the lattices of $\mathbb{L}$-fuzzy sets, generalized fuzzy sets and intuitionistic sets. The second paper in this series will extend this enquiry by introducing subtextures and quotient textures and a second series of papers is under preparation which consider topological aspects of this correspondence.

Keywords: Fuzzy lattices; *L*-fuzzy sets; Generalized fuzzy sets; Intuitionistic sets; Textures; Complemented textures; Product textures; Sums of textures (VARF)

1. Introduction

The notion of a ditopological texture space, under the name ditopological fuzzy structure, was introduced by the first author at the 2nd BUFSA Conference on Fuzzy Systems and Artificial Intelligence held at Trabzon University in 1992. It is a natural sequel to the work of the second author on the representation of fuzzy topologies by bitopological spaces [8–10]. However, in place of the full lattice of subsets of some base set S, attention is focused on a suitable sublattice of subsets, called a texturing of S. In this way, as will be confirmed in this paper, an exact point-set setting for the study of fuzzy sets is obtained.

In the context of textures, bitopologies are replaced by dichotomous topologies, or ditopologies for short. Some basic notions concerning these were presented in [2, 3]. In particular product textures were defined and the ditopological counterpart of joint compactness was introduced, and shown to be productive. Further results on ditopological compactness may be found in [4], and forms of ditopological paracompactness and connectedness are discussed in [5, 7], respectively.

The relation between textures and fuzzy sets was mentioned briefly in [2], and it is the aim of this paper to investigate this relation in detail. This will involve a discussion of complementation on product textures, and the notion of a sum of textures will be introduced to facilitate the representation of generalized fuzzy sets in the sense of [13], and to provide an alternative formulation of the representation of $\mathbb{L}$-fuzzy sets.

Our present concern is to develop the theory of texture spaces within a rigorous mathematical setting,

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but the potential for future applications is clear. Textures enable fuzzy sets to be studied as crisp subsets of some basic set and the operations of product, sum, sub-texture and quotient studied in this paper and its sequel place fuzzy sets in a new framework. Moreover, as we will see below, textures are strictly more general than fuzzy sets. This raises the exciting possibility of generalizing the notion of fuzzy set in ways that are mathematically advantageous and at the same time, of practical significance. Some ideas along these lines will be developed in a future paper.

For the benefit of the reader we will recall all necessary definitions and results from [2, 3]. [11] is an invaluable source of information about lattice theory.

Definition 1.1. Let S be a set. Then $\mathscr{S} \subseteq \mathscr{P}(S)$ is called a *texturing of* S, and S is said to be *textured by* $\mathscr{S}$. If

(1) $(\mathscr{S}, \subseteq)$ is a complete lattice containing S and $\emptyset$, and the meet and join operations in $(\mathscr{S}, \subseteq)$ are related with the intersection and union operations in $(\mathscr{P}(S), \subseteq)$ by the equalities

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i, \quad A_i \in \mathscr{S},\ i \in I,$$

for all index sets I, while

$$\bigvee_{i \in I} A_i = \bigcup_{i \in I} A_i, \quad A_i \in \mathscr{S},\ i \in I,$$

for all finite index sets I.

(2) $\mathscr{S}$ is completely distributive.

(3) $\mathscr{S}$ separates the points of S. That is, given $s_1 \neq s_2$ in S we have $A \in \mathscr{S}$ with $s_1 \in A$, $s_2 \notin A$, or $A \in \mathscr{S}$ with $s_2 \in A$, $s_1 \notin A$.

If S is textured by $\mathscr{S}$ we call $(S, \mathscr{S})$ a *texture space*, or simply a *texture* for short.

The mapping $s \to P_s = \bigcap\{A \in \mathscr{S} \mid s \in A\}$ is a natural embedding of S in $\mathscr{S}$. Recall that an element $M \neq \emptyset$ of $\mathscr{S}$ is a *molecule* if $M \subseteq A_1 \cup A_2 \Rightarrow M \subseteq A_1$ or $M \subseteq A_2$ for all $A_1, A_2 \in \mathscr{S}$. Clearly $\{P_s \mid s \in S\}$ is a set of molecules in $\mathscr{S}$ which is a *base for* $\mathscr{S}$ in the sense that

$$A = \bigvee_{s \in A} P_s = \bigcup_{s \in A} P_s$$

for all $A \in \mathscr{S}$.

We call $\mathscr{S}$ *simple* if all the molecules of $\mathscr{S}$ belong to $\{P_s \mid s \in S\}$.

A texturing $\mathscr{S}$ of S induces a partial ordering on S given by

$$s \leqslant t \Leftrightarrow P_s \subseteq P_t.$$

Clearly this is equivalent to $s \leqslant t \Leftrightarrow s \in P_t$.

A mapping $\sigma : \mathscr{S} \to \mathscr{S}$ is called a *complementation* if $\sigma^2(A) = A$ for all $A \in \mathscr{S}$ and $A \subseteq B$ in $\mathscr{S}$ implies $\sigma(B) \subseteq \sigma(A)$. A complementation is necessarily bijective. A texture with a complementation is said to be *complemented.*

Examples 1.2. (1) Clearly $(X, \mathscr{P}(X))$ is a texture, which represents the full set structure of X. Here $P_x = \{x\}$ and these are the only molecules so the texture is simple. The induced partial order is trivial. Clearly, $\pi = \pi_X : Y \subseteq X \mapsto X \setminus Y$ is a complementation on $(X, \mathscr{P}(X))$.

(2) Let $L = (0, 1]$ and $\mathscr{L} = \{(0, r] \mid r \in [0, 1]\}$. Then $(L, \mathscr{L})$ is a texture for which the join is not always the same as the union. The only molecules are $P_s = (0, s]$, $s \in S$, so the texture is simple. The induced partial order is the usual order on $(0, 1]$. Clearly $\lambda(0, r] = (0, 1 - r]$ is a natural complementation on $(L, \mathscr{L})$.

(3) Now let $L = [0, 1]$, and take $\mathscr{L} = \{[0, r] \mid r \in [0, 1]\} \cup \{[0, r) \mid r \in [0, 1]\}$. Then $(L, \mathscr{L})$ is a texture for which the join and union coincide. In addition to $P_r = [0, r]$, $r \in L$, the sets $[0, r)$ are also molecules, so this is not a simple texture. Again the induced order is the usual order on $[0, 1]$. A complementation λ may be defined by setting $\lambda[0, r] = [0, 1 - r)$ and $\lambda[0, r) = [0, 1 - r]$, $r \in Ł$.

(4) $\mathscr{S} = \{\emptyset, \{a, b\}, \{b\}, \{b, c\}, S\}$ is a simple texturing of $S = \{a, b, c\}$. Clearly $P_a = \{a, b\}$, $P_b = \{b\}$ and $P_c = \{b, c\}$ so the induced partial order is the reflexive closure of $b \leqslant a$, $b \leqslant c$. It is not possible to define a complementation on $(S, \mathscr{S})$.

We shall find the following properties of complementation useful in the sequel.

Lemma 1.3. *Let* $(S, \mathscr{S}, \sigma)$ *be a complemented texture. Then for all* $P \in \mathscr{S}$ *we have*

$$P = \bigcap_{t \in \sigma(P)} \sigma(P_t).$$

Proof. Take the image under σ of both sides of the identity $\sigma(P) = \bigvee_{t\in\sigma(P)} P_t$.

Corollary 1. $P_s = \bigcap_{t\in\sigma(P_s)} \sigma(P_t)$ *for all* $s \in S$.

Corollary 2. $t \in \sigma(P_s) \Leftrightarrow s \in \sigma(P_t)\ \forall s,\ t \in S$.

If $(S_i, \mathscr{S}_i)$, $i = 1,2$ are textures, a mapping $\theta : S_1 \to S_2$ is an *isomorphism* [4] if it is a bijection and the mapping $\theta : \mathscr{S}_1 \to \mathscr{S}_2$ defined by $\theta : A \mapsto \theta(A)$ is a bijection also. Clearly an isomorphism of textures preserves arbitrary meets and joins. If in addition, σ_i is a complementation on $(S_i, \mathscr{S}_i)$, $i = 1,2$, and θ satisfies the additional condition $\sigma_2 \circ \theta = \theta \circ \sigma_1$ then θ is an isomorphism of $(S_1, \mathscr{S}_1, \sigma_1)$ with $(S_2, \mathscr{S}_2, \sigma_2)$.

2. Representation of fuzzy lattices and L-fuzzy sets

Let $\mathbb{L}$ be a fuzzy lattice, i.e. a completely distributive lattice with order reversing involution $'$. Let L denote the set of molecules in $\mathbb{L}$ and set $\varphi(a) = \{m \in L \mid m \leqslant a\}$ and $\mathscr{L} = \{\varphi(a) \mid a \in \mathbb{L}\}$ for $a \in \mathbb{L}$. Then:

Theorem 2.1. $(L, \mathscr{L})$ *is a simple texture with complement* $\lambda(\varphi(a)) = \varphi(a')$, $a \in \mathbb{L}$, *and* $\varphi : \mathbb{L} \to \mathscr{L}$ *is a lattice isomorphism which preserves complementation.*

Conversely, every complemented simple texture may be obtained in this way from a suitable fuzzy lattice.

Proof. The fact that $\mathscr{L}$ is a complete lattice and $\varphi : \mathbb{L} \to \mathscr{L}$ is an isomorphism of complete lattices follows trivially from

$$a \leqslant b \Leftrightarrow \varphi(a) \subseteq \varphi(b) \quad \forall a, b \in \mathbb{L}$$

which holds by virtue of the fact that the molecules of $\mathbb{L}$ form a base (see [11], where this result is given in terms of the dual notion of prime ideal). Likewise

$$\bigcap_{\alpha\in A} \varphi(a_\alpha) = \bigwedge_{\alpha\in A} \varphi(a_\alpha)$$

for $a_\alpha \in \mathbb{L}$ and all index sets A is a consequence of

$$m \leqslant \bigwedge_{\alpha} a_\alpha \Leftrightarrow m \leqslant a_\alpha \quad \forall \alpha \in A$$

for $m \in L$, and

$$\bigcup_{\alpha\in A} \varphi(a_\alpha) = \bigvee_{\alpha\in A} \varphi(a_\alpha)$$

for finite A follows from

$$m \leqslant \bigvee_{\alpha} a_\alpha \Leftrightarrow m \leqslant a_\alpha \quad \text{for some } \alpha \in A$$

whenever $m \in L$. Since $L = \varphi(1) \in \mathscr{L}$ and $\emptyset = \varphi(0) \in \mathscr{L}$ we see that $(L, \mathscr{L})$ satisfies Definition 1.1(1). $\mathscr{L}$ is completely distributive since it is isomorphic to $\mathbb{L}$ and it separates points of L since if $m, n \in L$ satisfy $m \neq n$ then $m \nleqslant n$ or $n \nleqslant m$ so $m \notin \varphi(n)$ or $n \notin \varphi(m)$. Hence $(L, \mathscr{L})$ is a texture. Finally, if $\varphi(a)$ is a molecule in $\mathscr{L}$ it is easy to see that a is a molecule in $\mathbb{L}$, so all the molecules in $\mathscr{L}$ have the form $\varphi(m)$ for $m \in L$. However, for $m \in L$ we clearly have $P_m = \varphi(m)$ and so $(L, \mathscr{L})$ is simple. It is trivial that $\lambda(\varphi(a)) = \varphi(a')$ defines a complementation and that λ is preserved under φ.

Conversely, suppose that $(S, \mathscr{S}, \sigma)$ is a simple complemented texture. Then $\mathscr{S}$ is a fuzzy lattice with involution $'$ defined by $A' = \sigma(A)$, $A \in \mathscr{S}$. For $A \in \mathscr{S}$ let $\varphi(A) = \{P_s \mid P_s \subseteq A\} = \{P_s \mid s \in A\}$, $S^\star = \varphi(S) = \{P_s \mid s \in S\}$, $\sigma^\star(\varphi(A)) = \varphi(\sigma(A))$ and $\mathscr{S}^\star = \{\varphi(A) \mid A \in \mathscr{S}\}$, so that $(S^\star, \mathscr{S}^\star, \sigma^\star)$ is the complemented texture corresponding to $\mathscr{S}$. Clearly $\theta : s \mapsto P_s$ is an isomorphism between $(S, \mathscr{S}, \sigma)$ and $(S^\star, \mathscr{S}^\star, \sigma^\star)$. □

If $(L, \mathscr{L}, \lambda)$ is defined as above for the fuzzy lattice $\mathbb{L}$ we will call $(L, \mathscr{L})$ ($(L, \mathscr{L}, \lambda)$) the (*complemented*) *fuzzy texture of* $\mathbb{L}$.

By Theorem 2.1, all fuzzy textures are simple. This means that many potentially important textures, such as the texture of Examples 1.2(3), do not correspond to fuzzy lattices, i.e. textures are strictly more general than fuzzy lattices. Moreover, the property of being simple does not seem to play an essential role in the theory of textures, at least not for those results presented in the present paper and its sequel. This suggests that non-simple textures may provide a useful model on which to base definitions of generalized fuzzy sets with desirable properties. We will not pursue this line of inquiry further here, but as mentioned in the introduction some preliminary studies are underway which will be reported later.

Now let $\mathbb{L}$ be a fuzzy lattice and X a non-empty set. Then $\mathbb{W}=\mathbb{L}^X$, the set of $\mathbb{L}$-fuzzy sets [12] on X, is also a fuzzy lattice under the point-wise ordering $f\leqslant g \Leftrightarrow f(x)\leqslant g(x)\ \forall x\in X$, and the involution $f'(x)=f(x)'$. We will refer to the fuzzy texture $(W,\mathscr{W},\omega)$ of $\mathbb{W}$ as the *complemented* $\mathbb{L}$*-fuzzy texture on* X. The elements of W are just the "fuzzy points" of $\mathbb{W}$, i.e. the functions

$$x_m(z)=\begin{cases} m & \text{if } z=x,\\ 0 & \text{otherwise,}\end{cases}$$

for $x\in X$ and $m\in L$, where as before L is the set of molecules of $\mathbb{L}$. Representing x_m by the pair (x,m) we obtain a canonical form for $(W,\mathscr{W},\omega)$ by setting $W=X\times L$, and then

$$\mathscr{W}=\{\varphi(f)\mid f\in\mathbb{W}\},$$

$$\varphi(f)=\{(x,m)\in W\mid x_m\leqslant f\}$$

$$=\{(x,m)\in W\mid m\leqslant f(x)\}$$

while

$$\omega(\varphi(f))=\varphi(f')=\{(x,m)\mid m\leqslant f(x)'\}$$

gives the complement.

We wish to show that $(W,\mathscr{W})$ is actually the product of the textures $(X,\mathscr{P}(X))$ and $(L,\mathscr{L})$. First let us recall from [3] the definition of the product of texture spaces. Let $(S_i,\mathscr{S}_i)$, $i\in I$ be textures, set $S=\prod_{i\in I}S_i$ and for $k\in I$ and $A\subseteq S_k$ let

$$E(k,A)=\prod_{i\in I}Y_i,$$

where

$$Y_i=\begin{cases} A & \text{if } i=k,\\ S_i & \text{otherwise.}\end{cases}$$

Then the *product* of the texturings $\mathscr{S}_i$, $i\in I$, is the texturing $\mathscr{S}$ of S which consists of arbitrary intersections of elements of the set

$$\mathscr{E}=\left\{\bigcup_{j\in J}E(j,A_j)\ \middle|\ J\subseteq I,\ A_j\in\mathscr{S}_j\right\}.$$

It is shown in [3] that this does indeed define a texturing in the sense of Definition 1.1. The following results from [3] will prove useful later on.

Lemma 2.2. 1. *If* $A_i\in\mathscr{S}_i$, $i\in I$, *then* $\prod_{i\in I}A_i\in\mathscr{S}$.

2. *For* $s=(s_i)\in S$ *we have* $P_s=\prod_{i\in I}P_{s_i}=\bigcap_{i\in I}E(i,P_{s_i})$.

Lemma 2.3. *For* $\delta\in D$ *suppose that* $J_\delta\subseteq I$ *and for* $j\in J_\delta$ *let* $A_j^\delta\in\mathscr{S}_j$. *Then setting*

$$E_\delta=\bigcup_{j\in J_\delta}E(j,A_j^\delta)\in\mathscr{E}$$

for $\delta\in D$ *we have*

$$\bigvee_{\delta\in D}E_\delta=\bigcup_{j\in\cup_\delta J_\delta}E\left(j,\bigvee\{A_j^\delta\mid j\in J_\delta\}\right).$$

We may now state:

Lemma 2.4. *The* $\mathbb{L}$*-fuzzy texture* $(W,\mathscr{W})$ *on* X *is the product of the set structure* $(X,\mathscr{P}(X))$ *of* X *and the fuzzy texture* $(L,\mathscr{L})$ *of* $\mathbb{L}$.

Proof. By definition, each element of the product texturing of $X\times L$ may be written as an intersection of sets of the form

$$(Y\times L)\cup(X\times\varphi(a))$$

for $Y\subseteq X$ and $a\in\mathbb{L}$.

To show the product texturing is coarser than $\mathscr{W}$ it will be sufficient to show $(Y\times L)\cup(X\times\varphi(a))\in\mathscr{W}$. However, this is trivial for if we define $f:X\to\mathbb{L}$ by

$$f(x)=\begin{cases} 1 & \text{for } x\in Y,\\ a & \text{for } x\in X\setminus Y,\end{cases}$$

then clearly $\varphi(f)=(Y\times L)\cup(X\times\varphi(a))$.

Conversely, to show $\mathscr{W}$ is coarser than the product texturing, it suffices to take $f\in\mathbb{W}$ and note that

$$\varphi(f)=\bigcap_{z\in X}[((X\setminus\{z\})\times L)\cup(X\times\varphi(f(z)))],$$

while this set belongs to the product texturing. $\square$

Complementation on products was not considered in [2, 3]. It will be appropriate to do so now.

Theorem 2.5. *Let* $(S_i,\mathscr{S}_i,\sigma_i)$, $i\in I$, *be complemented textures and* $(S,\mathscr{S})$ *their product. For* $A\in\mathscr{S}$ *define*

$$\sigma(A)=\bigcap_{s\in A}\bigcup_{i\in I}E(i,\sigma_i(P_{s_i})).$$

Then σ *is a complementation on* $(S,\mathscr{L})$.

Proof. That σ is a mapping on $\mathscr{S}$ satisfying $A\subseteq B\Rightarrow\sigma(B)\subseteq\sigma(A)$ for all $A, B\in\mathscr{S}$ is trivial, so it remains to show $\sigma(\sigma(A))=A$ for $A\in\mathscr{S}$.

Take $s\in A$. Then for $t\in\sigma(A)$ we have $t\in\bigcup_{i\in I}E(i,\sigma_i(P_{s_i}))$ and hence there exists $i=i(t)\in I$ with $t_i\in\sigma_i(P_{s_i})$. By Corollary 2 of Lemma 1.3 we deduce $s_i\in\sigma_i(P_{t_i})$ for this i, whence $s\in\sigma(\sigma(A))=\bigcap_{t\in\sigma(A)}\bigcup_{i\in I}E(i,\sigma_i(P_{t_i}))$. This establishes $A\subseteq\sigma(\sigma(A))$.

Now take $r\in\sigma(\sigma(A))$ and suppose that $r\notin A$. We may write

$$A=\bigvee_{s\in A}P_s=\bigvee_{s\in A}\bigcap_{i\in I}E(i,P_{s_i})=\bigcap_{\gamma\in I^A}\bigvee_{s\in A}E(\gamma(s),P_{s_{\gamma(s)}})$$

by the complete distributivity of $\mathscr{S}$. For each $\gamma\in I^A$ we may now apply Lemma 2.3 with $D=A$, $J_s=\{\gamma(s)\}$ for $s\in A$ and $A_j^s=P_{s_j}$ for $j\in J_s$ to give

$$A=\bigcap_{\gamma\in I^A}\bigcup_{j\in\gamma(A)}E\left(j,\bigvee\{P_{s_j}\mid s\in A,\ \gamma(s)=j\}\right).$$

Since $r=(r_i)\notin A$ there exists $\gamma\in I^A$ so that for all $j\in\gamma(A)$ we have $r_j\notin\bigvee\{P_{s_j}\mid s\in A,\ \gamma(s)=j\}$. If for each $j\in\gamma(A)$ we apply Lemma 1.2 to $P=\bigvee\{P_{s_j}\mid s\in A,\ \gamma(s)=j\}$ we see that there exists $t_j\in\sigma_j(\bigvee\{P_{s_j}\mid s\in A,\ \gamma(s)=j\})=\bigcap\{\sigma_j(P_{s_j})\mid s\in A,\ \gamma(s)=j\}$ satisfying $r_j\notin\sigma_j(P_{t_j})$. For $i\notin\gamma(A)$ we may choose $t_i\in S_i\setminus\sigma_i(P_{r_i})$ since $r_i\in P_{r_i}\neq\emptyset\Rightarrow\sigma_i(P_{s_i})\neq S_i$, whence $r_i\notin\sigma_i(P_{t_i})$ by Lemma 1.3, Corollary 2. Let $t=(t_i)$. Then $t\in\sigma(A)$. To see this we need only note that by definition $\sigma(A)=\bigcap_{s\in A}\bigcup_{i\in I}E(i,\sigma_i(P_{s_i}))$ so by the complete distributivity of the lattice $\mathscr{P}(S)$

$$\sigma(A)=\bigcup_{\gamma\in I^A}\bigcap_{s\in A}E(\gamma(s),\sigma_{\gamma(s)}(P_{s_{\gamma(s)}}))$$

and for the γ found above $t\in E(\gamma(s),\sigma_{\gamma(s)}(P_{s_{\gamma(s)}}))$ for each $s\in A$. Now

$$r\in\sigma(\sigma(A))=\bigcap_{u\in\sigma(A)}\bigcup_{i\in I}E(i,\sigma_i(P_{u_i}))$$

so $r\in\bigcup_{i\in I}E(i,\sigma_i(P_{t_i}))$ since $t\in\sigma(A)$. However this is a contradiction since by the definition of t we have $r_i\notin\sigma_i(P_{t_i})$ for all $i\in I$. Hence $r\in A$ and we have established $\sigma(\sigma(A))\subseteq A$. Thus σ is indeed a complementation on the product texture $(S,\mathscr{S})$. □

Definition 2.6. The complementation σ defined on the product $(S,\mathscr{S})$ of the complemented textures $(S_i,\mathscr{S}_i,\sigma_i)$, $i\in I$ by $\sigma(A)=\bigcap_{s\in A}\bigcup_{i\in I}E(i,\sigma_i(P_{s_i}))$ for all $A\in\mathscr{S}$ is called the *product* of the complementations σ_i. $(S,\mathscr{S},\sigma)$ will be called the *complemented product* of $(S_i,\mathscr{S}_i,\sigma_i)$, $i\in I$.

We may note the following. The proof follows much the same lines as the proof of Theorem 2.5, and is omitted.

Lemma 2.7. *If $(S,\mathscr{S},\sigma)$ is the complemented product of $(S_i,\mathscr{S}_i,\sigma_i)$, $i\in I$, then for $i\in I$ and $A_i\in\mathscr{S}_i$ we have*

$$\sigma(E(i,A_i))=E(i,\sigma_i(A_i)).$$

Corollary. *For $s\in S=\prod_{i\in I}S_i$ we have*

$$\sigma(P_s)=\bigcup_{i\in I}E(i,\sigma_i(P_{s_i})).$$

Proof. $\sigma(P_s)=\sigma(\prod_{i\in I}P_{s_i})=\sigma(\bigcap_{i\in I}E(i,P_{s_i}))=\bigvee_{i\in I}E(i,\sigma_i(P_{s_i}))$ by Lemma 2.7. Since $\bigcup_{i\in I}E(i,\sigma_i(P_{s_i}))\in\mathscr{S}$ we obtain $\sigma(P_s)=\bigcup_{i\in I}E(i,\sigma_i(P_{s_i}))$. □

We are now in a position to compare the complementation ω on $(W,\mathscr{W})$ with the complementations on $(X,\mathscr{P}(X))$ and $(L,\mathscr{L})$.

Lemma 2.8. *The complementation ω on $(W,\mathscr{W})$ is the product of the complementations π on $(X,\mathscr{P}(X))$ and λ on $(L,\mathscr{L})$.*

Proof. It will be sufficient to show that the effect of the two complementations on sets of the form $(Y\times L)\cup(X\times\varphi(a))$, $Y\subseteq X$, $a\in\mathbb{L}$ is the same. By Lemma 2.7 the complement of this set for the product complementation σ is $\sigma((Y\times L)\cup(X\times\varphi(a)))=\sigma(Y\times L)\cap\sigma(X\times\varphi(a))=(\pi(Y)\times L)\cap(X\times\lambda(\varphi(a)))=((X\setminus Y)\times L)\cap(X\times\varphi(a'))=(X\setminus Y)\times\varphi(a')$. On the other hand $(Y\times L)\cup(X\times\varphi(a))=\varphi(f)$, where $f:X\to\mathbb{L}$ is defined by

$$f(x)=\begin{cases}1 & \text{for } x\in Y,\\ a & \text{for } x\in X\setminus Y,\end{cases}$$

so $\omega(\varphi(f))=\varphi(f')$ where

$$f'(x)=\begin{cases}0 & \text{for } x\in Y,\\ a' & \text{for } x\in X\setminus Y.\end{cases}$$

It is clear that $\varphi(f')=(X\setminus Y)\times\varphi(a')$ as required. □

Combining Lemmas 2.4 and 2.8 gives:

Theorem 2.9. *The complemented* $\mathbb{L}$*-fuzzy texture on* X *is the complemented product of* $(X, \mathscr{P}(X), \pi)$ *and* $(L, \mathscr{L}, \lambda)$.

Examples 2.10. (1) For classic fuzzy sets on X we have $\mathbb{L} = [0,1]$. Every non-zero element of $\mathbb{L}$ is a molecule, so $L = (0,1]$ and $\mathscr{L} = \{(0,r] \mid 0 \leqslant r \leqslant 1\}$. Hence $W = X \times (0,1]$ and the sets of $\mathscr{W}$ are intersections of sets of the form $(Y \times (0,1]) \cup (X \times (0,r])$, $Y \subseteq X$ and $0 \leqslant r \leqslant 1$. Moreover $\omega((Y \times (0,1]) \cup (X \times (0,r])) = (X \backslash Y) \times (0, 1-r]$.

(2) Take $\mathbb{L} = \{0, \frac{1}{2}, 1\}$. Then $L = \{\frac{1}{2}, 1\}$ and $\mathscr{L} = \{\emptyset, \{\frac{1}{2}\}, L\}$. Hence in this case $W = X \times \{\frac{1}{2}, 1\}$ and the elements of $\mathscr{W}$ are intersections of sets of the form $X \times L$, $Y \times \{\frac{1}{2}, 1\}$ and $(Y \times \{\frac{1}{2}, 1\}) \cup (X \times \{\frac{1}{2}\})$, $Y \subseteq X$, whose complements are respectively $\emptyset$, $(X \backslash Y) \times \{\frac{1}{2}, 1\}$ and $(X \backslash Y) \times \{\frac{1}{2}\}$.

We end this section with a further characterization of intuitionistic textures. In [1] Atanassov introduced the notion of intuitionistic fuzzy set, and the crisp version of these sets, intuitionistic sets, was given by Çoker in [6]. In [4] intuitionistic textures were defined as those textures which correspond to intuitionistic sets, and several characterizations were given. We now show that intuitionistic textures are precisely the textures described in the last example, so showing that intuitionistic sets are essentially $\mathbb{L}$-fuzzy sets, where $\mathbb{L}$ is the fuzzy lattice $\{0, \frac{1}{2}, 1\}$.

Theorem 2.11. $(S, \mathscr{S}, \sigma)$ *is an intuitionistic texture if and only if it is isomorphic to the texture* $(W, \mathscr{W}, \omega)$ *corresponding to the* $\mathbb{L}$*-fuzzy sets on some set* X *for* $\mathbb{L} = \{0, \frac{1}{2}, 1\}$.

Proof. We prove sufficiency, leaving the proof of necessity to the interested reader. Let $(W, \mathscr{W}, \omega)$ be a texture as in Examples 2.10(2). We verify the conditions of [4, Theorem 2.5(iv)]. $W = X \times L = X \times \{\frac{1}{2}, 1\} = (X \times \{\frac{1}{2}\}) \cup (X \times \{1\})$. Let $T = X \times \{\frac{1}{2}\}$ and define $f : T \to W \backslash T = X \times \{1\}$ by $(x, \frac{1}{2}) \mapsto (x, 1)$, $x \in X$. Clearly f is injective. For $A \in \mathscr{W}$ let $A_1 = A \cap T$ and $A_2 = A \cap (W \backslash T)$. Then it is easy to verify that

(1) $A = A_1 \cup A_2$, $A_1 \subseteq T$ and $A_2 \subseteq f(A_1)$,
(2) $\omega(A) = W \backslash (f(A_1) \cup f^{-1}(A_2))$.

Conversely, any subset of W satisfying (1) and (2) belongs to $\mathscr{W}$, whence the result follows by [4, Theorem 2.5 (iv)]. □

3. Sums of texture spaces – the representation of generalized fuzzy sets

In order to represent generalized fuzzy sets in the sense of Nakajima [13] we now introduce the notion of the sum of texture spaces. This will, in particular, lead to an alternative interpretation of the representation of $\mathbb{L}$-fuzzy sets given in Theorem 2.9.

Lemma 3.1. *Let* $(S_i, \mathscr{S}_i)$, $i \in I$, *be textures with* $S_i \cap S_j = \emptyset$ *for* $i \neq j$. *Let* $S = \bigcup_{i \in I} S_i$ *and* $\mathscr{S} = \{A \mid A \subseteq S, A \cap S_i \in \mathscr{S}_i \ \forall i \in I\}$. *Then* $(S, \mathscr{S})$ *is a texture. Further, if* σ_i *is a complementation on* $(S_i, \mathscr{S}_i)$, $i \in I$, *then* σ *defined on* $\mathscr{S}$ *by*

$$\sigma(A) \cap S_i = \sigma_i(A \cap S_i), \quad i \in I,$$

makes $(S, \mathscr{S}, \sigma)$ *a complemented texture.*

Proof. $S, \emptyset \in \mathscr{S}$ follow trivially from the definition. Moreover, for $A_\alpha \in \mathscr{S}$ we have $(\bigcap_\alpha A_\alpha) \cap S_i = \bigcap_\alpha (A_\alpha \cap S_i) \in \mathscr{S}_i$ for each $i \in I$, so $\bigcap_\alpha A_\alpha \in \mathscr{S}$. Thus $\mathscr{S}$ is a complete lattice and $\bigwedge_\alpha A_\alpha = \bigcap_\alpha A_\alpha$. Now let us verify that

$$\left(\bigvee_\alpha A_\alpha \right) \cap S_i = \bigvee_\alpha (A_\alpha \cap S_i) \quad \forall i \in I.$$

Clearly,

$$\left(\bigvee_\alpha A_\alpha \right) \cap S_i = \bigcap \left\{ A \,\middle|\, A \in \mathscr{S}, \bigcup_\alpha A_\alpha \subseteq A \right\} \cap S_i$$
$$= \bigcap \left\{ A \cap S_i \,\middle|\, A \in \mathscr{S}, \bigcup_\alpha A_\alpha \subseteq A \right\}.$$

Now $A \in \mathscr{S}$, $\bigcup_\alpha A_\alpha \subseteq A \Rightarrow A \cap S_i \in \mathscr{S}_i$, $\bigcup_\alpha (A_\alpha \cap S_i) \subseteq A \cap S_i$ so

$$\bigvee_\alpha (A_\alpha \cap S_i) = \bigcap \left\{ A_i \,\middle|\, A_i \in \mathscr{S}_i, \bigcup_\alpha (A_\alpha \cap S_i) \subseteq A_i \right\}$$
$$\subseteq \left(\bigvee_\alpha A_\alpha \right) \cap S_i.$$

On the other hand if $A_i \in \mathcal{S}_i$ and $\bigcup_\alpha (A_\alpha \cap S_i) \subseteq A_i$, let $A = \bigcup_{j\neq i} S_j \cup A_i$. Then $A \in \mathcal{S}$, $\bigcup_\alpha A_\alpha \subseteq A$ and $A \cap S_i = A_i$, whence $(\bigvee_\alpha A_\alpha) \cap S_i \subseteq \bigvee_\alpha (A_\alpha \cap S_i)$ as required.

Now for $A_1, A_2, \ldots, A_n \in \mathcal{S}$ and $i \in I$ we have $(\bigvee_{k=1}^n A_k) \cap S_i = \bigvee_{k=1}^n (A_k \cap S_i) = \bigcup_{k=1}^n (A_k \cap S_i) = (\bigcup_{k=1}^n A_k) \cap S_i$ since $\mathcal{S}_i$ is a texturing. Hence

$$\bigvee_{k=1}^n A_k = \bigcup_{k=1}^n A_k$$

so $(S, \mathcal{S})$ satisfies Definition 1.1 (1). To show $(S, \mathcal{S})$ is completely distributive, for $\alpha \in A$, $\beta \in B_\alpha$, take $A_\beta^\alpha \in \mathcal{S}$. Then for any $i \in I$, $(\bigvee_{\alpha\in A} \bigcap_{\beta \in B_\alpha} A_\beta^\alpha) \cap S_i = \bigvee_{\alpha \in A}((\bigcap_{\beta\in B_\alpha} A_\beta^\alpha) \cap S_i) = \bigvee_{\alpha\in A} \bigcap_{\beta\in B_\alpha} (A_\beta^\alpha \cap S_i) = \bigcap_{\gamma \in \Pi B_\alpha} \bigvee_{\alpha\in A} (A^\alpha_{\gamma(\alpha)} \cap S_i) = \bigcap_{\gamma\in\Pi B_\alpha}((\bigvee_{\alpha\in A} A^\alpha_{\gamma(\alpha)}) \cap S_i) = (\bigcap_{\gamma\in\Pi B_\alpha} \bigvee_{\alpha \in A} A^\alpha_{\gamma(\alpha)}) \cap S_i$ using the above and the complete distributivity of $(S_i, \mathcal{S}_i)$. Thus

$$\bigvee_{\alpha\in A} \bigcap_{\beta \in B_\alpha} A_\beta^\alpha = \bigcap_{\gamma\in\Pi B_\alpha} \bigvee_{\alpha\in A} A^\alpha_{\gamma(\alpha)}$$

as required. Moreover, since $\mathcal{S}_i \subseteq \mathcal{S}$ for each $i \in I$ it is easy to see that $\mathcal{S}$ separates the points of S, whence $(S, \mathcal{S})$ is a texture.

Now suppose σ_i is a complementation on $(S_i, \mathcal{S}_i)$ and for $A \in \mathcal{S}$ define $\sigma(A)$ by

$$\sigma(A) \cap S_i = \sigma_i(A \cap S_i), \quad i \in I.$$

Clearly $\sigma(A) \in \mathcal{S}$ and for $A, B \in \mathcal{S}$ with $A \subseteq B$ we have $\sigma(B) \subseteq \sigma(A)$. Finally

$$\sigma(\sigma(A)) \cap S_i = \sigma_i(\sigma(A) \cap S_i) = \sigma_i(\sigma_i(A \cap S_i)) = A \cap S_i$$

for all $i \in I$, whence $\sigma(\sigma(A)) = A$. This completes the proof that σ is a complementation on $(S, \mathcal{S})$. □

Definition 3.2. The texture $(S, \mathcal{S})$ defined as in Lemma 3.1 is called the *sum* of the disjoint textures $(S_i, \mathcal{S}_i)$, $i \in I$. If σ_i is a complementation on $(S_i, \mathcal{S}_i)$ and σ is defined as in Lemma 3.1 then $(S, \mathcal{S}, \sigma)$ is the *complemented sum* of the $(S_i, \mathcal{S}_i, \sigma_i)$, $i \in I$.

Now let X be a set and for each $x \in X$ let $\mathbb{L}_x$ be a fuzzy lattice. Then functions f on X satisfying $f(x) \in \mathbb{L}_x$ for each $x \in X$ are called generalized fuzzy sets in [13]. Clearly these form the family

$$\mathbb{L} = \prod_{x\in X} \mathbb{L}_x = \left\{ f \,\middle|\, f : X \to \bigcup_{x\in X} \mathbb{L}_x,\ f(x) \in \mathbb{L}_x \right\},$$

which is a fuzzy lattice under the point-wise order $f \leqslant g \Leftrightarrow f(x) \leqslant g(x)\ \forall x \in X$ and involution $f'(x) = (f(x))'$, $x \in X$.

Denote by $(L, \mathcal{L}, \lambda)$ the complemented fuzzy texture of $\mathbb{L}$. We wish to show that $(L, \mathcal{L}, \lambda)$ is isomorphic in a natural way to a complemented sum of textures.

Let $(L_x, \mathcal{L}_x, \lambda_x)$ be the complemented fuzzy texture of $\mathbb{L}_x$, $x \in X$, and note first that the molecules in $\mathbb{L}$ have the form $f_{(x,m)}$ for $x \in X$ and $m \in L_x$, where

$$f_{(x,m)}(y) = \begin{cases} m & \text{if } y = x \\ 0_y & \text{otherwise,} \end{cases}$$

0_y being the smallest element of $\mathbb{L}_y$. Now let $(N_x, \mathcal{N}_x, \nu_x)$ be the complemented product of $(\{x\}, \mathcal{P}(\{x\}), \pi_{\{x\}})$ and $(L_x, \mathcal{L}_x, \lambda_x)$ and define

$$N = \bigcup_{x\in X} N_x = \bigcup_{x\in X} (\{x\} \times L_x).$$

Since $N_x \cap N_y = \emptyset$ for $x \neq y$ we may consider the complemented sum $(N, \mathcal{N}, \nu)$ of the textures $(N_x, \mathcal{N}_x, \nu_x)$. Then

Theorem 3.3. *The complemented texture* $(L, \mathcal{L}, \lambda)$ *corresponding to a lattice of generalized fuzzy sets on* X *is isomorphic to the complemented sum* $(N, \mathcal{N}, \nu)$ *of the textures* $(N_x, \mathcal{N}_x, \nu_x)$ *defined above.*

Proof. Define $\theta : N \to L$ by $(x, m) \mapsto f_{(x,m)}$. Clearly θ is a bijection. We show it is an isomorphism between $(N, \mathcal{N})$ and $(L, \mathcal{L})$. Take $A \in \mathcal{N}$. For each $x \in X$ we have $A \cap N_x \in \mathcal{N}_x$ so there exists $a_x \in \mathbb{L}_x$, necessarily unique, so that $A \cap N_x = \{x\} \times \varphi(a_x)$. Define $f \in \mathbb{L}$ by $f(x) = a_x\ \forall x \in X$. Then $\theta(A) = \varphi(f) \in \mathcal{L}$, and it is now trivial to verify that θ, regarded as a mapping from $\mathcal{N}$ to $\mathcal{L}$, is one to one and onto. Hence θ is indeed an isomorphism of textures. It remains to show that it preserves complementation. For $A \in \mathcal{N}$ and a_x, f defined as above we have $\nu(A) \cap N_x = \nu_x(A \cap N_x) = \{x\} \times \lambda_x(\varphi(a_x)) = \{x\} \times \varphi(a_x')$, using Lemma 2.7. Since $f'(x) = a_x'$, $\forall x \in X$, we have

$$\theta(\nu(A)) = \varphi(f') = \lambda(\varphi(f)) = \lambda(\theta(A)),$$

which is the required equality. □

Note. Nakajima only requires that the lattices $\mathbb{L}_x$ be complete Heyting Algebras. We could weaken the requirement that textures be completely distributive in order to represent Nakajima's generalized fuzzy sets in the general case. However, complete distributivity will prove invaluable when we come to discuss topological properties of textures, and we prefer not to make this change at the present time.

In the case where $\mathbb{L}_x = \mathbb{L}\ \forall x \in X$, the generalized fuzzy sets on X reduce to $\mathbb{L}$-fuzzy sets on X and then the sum $(N, \mathscr{N}, \nu)$ of $(N_x, \mathscr{N}_x, \nu_x)$, $N_x = \{x\} \times L$, provides an alternative representation of the fuzzy texture of $\mathbb{L}$. Indeed it is trivial to verify directly that $(W, \mathscr{W}, \omega)$ is isomorphic to $(N, \mathscr{N}, \nu)$ under the mapping

$$(x, m) \in W = X \times L \mapsto (x, m) \in N_x \subseteq N.$$

Now let us briefly consider the converse situation. Given a complemented simple texture $(S, \mathscr{S}, \sigma)$, under what conditions will it correspond to a lattice of (generalized) fuzzy sets on some set X? As it stands the answer is trivial – by Theorem 2.1 any complemented simple texture will correspond to the lattice of $\mathbb{L}$-fuzzy sets on a single point set X. To obtain a more useful result we restrict our attention to generalized fuzzy sets where the fuzzy lattices $\mathbb{L}_x$ are product irreducible, i.e. cannot be expressed in a non-trivial way as a product of fuzzy lattices. We refer to these as *irreducible* generalized fuzzy sets for short. We shall require the following:

Definition 3.4. Let $(S, \mathscr{S}, \sigma)$ be a complemented texture. Then $A \in \mathscr{S}$ is called *σ-irreducible* if whenever $A \subseteq A_1 \cup A_2$, $A_1, A_2 \in \mathscr{S}$, $B \cap A_k = C \cap A_k$ for $B, C \in \mathscr{S}$ implies $\sigma(B) \cap A_k = \sigma(C) \cap A_k$, $k = 1, 2$ and $A \cap A_1 \neq \emptyset \neq A \cap A_2$ we have $A \cap A_1 \cap A_2 \neq \emptyset$. $A \in \mathscr{S}$ is *σ-reducible* if it is not σ-irreducible. The *σ-component of* $s \in S$ is the set

$$K_\sigma(s) = \bigcup \{A \mid s \in A \in \mathscr{S},\ A \text{ is } \sigma\text{-irreducible}\}.$$

Lemma 3.5. *For each* $s \in S$, $s \in K_\sigma(s) \in \mathscr{S}$, *while any two σ-components are either equal or disjoint.*

Proof. Since P_s is clearly σ-irreducible we have $s \in P_s \subseteq K_\sigma(s) \subseteq \bigvee \{A \mid s \in A \in \mathscr{S},\ A \text{ is } \sigma\text{-}\mathit{irreducible}\} \in \mathscr{S}$. It is easy to show that this last set is σ-irreducible, whence $K_\sigma(s) = \bigvee \{A \mid s \in A \in \mathscr{S},\ A \text{ is } \sigma\text{-irreducible}\} \in \mathscr{S}$ as required. The remaining properties are immediate.

Lemma 3.6. *Let* $\mathbb{L}$ *be a fuzzy lattice and* $(L, \mathscr{L}, \lambda)$ *the complemented fuzzy texture of* $\mathbb{L}$. *Then* $\mathbb{L}$ *is isomorphic to a product of non-trivial fuzzy lattices* $\mathbb{L}_1$, $\mathbb{L}_2$ *if and only if* L *is λ-reducible in* $(L, \mathscr{L}, \lambda)$.

Proof. $\Rightarrow$. If $\mathbb{L}$ is isomorphic to $\mathbb{L}_1 \times \mathbb{L}_2 = \{f : \{1, 2\} \to \mathbb{L}_1 \cup \mathbb{L}_2,\ f(k) \in \mathbb{L}_k,\ k = 1, 2\}$ then by Theorem 3.3, $(L, \mathscr{L}, \lambda)$ is the complemented sum of $(N_1, \mathscr{N}_1, \nu_1)$ and $(N_2, \mathscr{N}_2, \nu_2)$, where $N_k = \{k\} \times L_k$, $k = 1, 2$. Clearly the existence of these sets implies that $L = N_1 \cup N_2$ is λ-reducible in $(L, \mathscr{L}, \lambda)$.

$\Leftarrow$. Let L be λ-reducible in $(L, \mathscr{L}, \lambda)$. Then we have $L_1, L_2 \in \mathscr{L}$ with $L = L_1 \cup L_2$, for $A, B \in \mathscr{L}$, $A \cap L_k = B \cap L_k \Rightarrow \lambda(A) \cap L_k = \lambda(B) \cap L_k$, $k = 1, 2$, $L_1 \neq \emptyset \neq L_2$ but $L_1 \cap L_2 = \emptyset$. $\mathscr{L}_k = \{L_k \cap A \mid A \in \mathscr{L}\}$ is a fuzzy lattice on L_k, $k = 1, 2$. Moreover a complement λ_k on L_k is well defined by $\lambda_k(L_k \cap A) = \lambda(A) \cap L_k$ in view of the conditions on L_k, $k = 1, 2$. It is clear that $(L, \mathscr{L}, \lambda)$ is the complemented sum of $(L_1, \mathscr{L}_1, \lambda_1)$ and $(L_2, \mathscr{L}_2, \lambda_2)$, whence if $\mathbb{L}_1$, $\mathbb{L}_2$ are the corresponding fuzzy lattices in the sense of Theorem 2.1, $(L, \mathscr{L}, \lambda)$ corresponds to $\mathbb{L}_1 \times \mathbb{L}_2$. Hence $\mathbb{L}$ is isomorphic to a non-trivial product of fuzzy lattices, as required. □

We may now state

Theorem 3.7. *The complemented texture* $(S, \mathscr{S}, \sigma)$ *corresponds to a lattice of irreducible generalized fuzzy sets if and only if*

(1) *Each* $K_\sigma(s)$, $s \in S$, *satisfies*

$$A, B \in \mathscr{S},\ A \cap K_\sigma(s) = B \cap K_\sigma(s) \Rightarrow \sigma(A) \cap K_\sigma(s) = \sigma(B) \cap K_\sigma(s),$$

and:

(2) *Arbitrary unions of the components* $K_\sigma(s)$, $s \in S$, *belong to* $\mathscr{S}$.

Proof. $\Rightarrow$. Suppose $(S, \mathscr{S}, \sigma)$ corresponds to the lattice $\prod_{x \in X} \mathbb{L}_x$ of irreducible generalized fuzzy sets. By Theorem 3.3, $(S, \mathscr{S}, \sigma)$ may be expressed as a sum of the structures $(N_x, \mathscr{N}_x, \nu_x)$. By Lemma 3.6 each N_x is ν_x-irreducible in $(N_x, \mathscr{N}_x, \nu_x)$, and hence σ-irreducible in $(S, \mathscr{S}, \sigma)$. Since these sets form a partition we see that they are the σ-components, and conditions (1) and (2) are now easily verified.

$\Leftarrow$. Choose $X \subseteq S$ with the following properties:
(a) $\bigcup_{x\in X} K_\sigma(x) = S$, and
(b) $x_1, x_2 \in X$, $x_1 \neq x_2 \Rightarrow K_\sigma(x_1) \cap K_\sigma(x_2) = \emptyset$.

For each $x \in X$ let $S_x = K_\sigma(x)$, $\mathscr{S}_x = \{S_x \cap A \mid A \in \mathscr{S}\}$ and $\sigma_x(A \cap S_x) = \sigma(A) \cap S_x$, $A \in \mathscr{S}$, which is well defined by (1). It is easy to verify that $(S, \mathscr{S}, \sigma)$ is the complemented sum of the structures $(S_x, \mathscr{S}_x, \sigma_x)$, $x \in X$. Moreover, by Lemma 3.6 and Theorem 2.1 each $(S_x, \mathscr{S}_x, \sigma_x)$ corresponds to a product irreducible fuzzy lattice $\mathbb{L}_x$, whence $(S, \mathscr{S}, \sigma)$ corresponds to $\prod_{x\in X} \mathbb{L}_x$, as required. □

Corollary. *$(S, \mathscr{S}, \sigma)$ is isomorphic to a lattice of $\mathbb{L}$-fuzzy sets with $\mathbb{L}$ product irreducible if and only if it satisfies the conditions of Theorem 3.7 and the textures on $K_\sigma(s)$, $s \in S$, are isomorphic.*

In particular instances it may be possible to choose a distinguished representative in each component, so obtaining a canonical copy of X in S.

Example 3.8. Let S be the punctured unit disc and $\mathscr{S}$ the set of all subsets A of S satisfying
(i) $(r,\theta) \in A$, $0 < s \leqslant r \Rightarrow (s,\theta) \in S$, and
(ii) $(r_i,\theta) \in A$, $i \in I \Rightarrow (\sup r_i, \theta) \in A$.

Clearly $(S, \mathscr{S})$ is a texture space, and a complementation σ may be defined by $\sup\{r \mid (r,\theta) \in \sigma(A)\} = 1 - \sup\{r \mid (r,\theta) \in A\}$. The reader may easily verify that for $(s,\theta) \in S$ we have $K_\sigma(s,\theta) = \{(r,\theta) \mid 0 < r \leqslant 1\}$. Indeed, $(S, \mathscr{S}, \sigma)$ is isomorphic to the complemented fuzzy texture of the lattice of fuzzy subsets of $X = [0, 2\pi)$. X may be embedded canonically in S as, for example, the circumference of the disc.

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