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Plain ditopological texture spaces

İsmail U. Tiryaki [a,*], Lawrence M. Brown [b]

[a] Abant İzzet Baysal University, Faculty of Science and Letters, Department of Mathematics, Bolu, Turkey
[b] Hacettepe University, Faculty of Science, Mathematics Department, 06800 Beytepe, Ankara, Turkey

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ABSTRACT

Using the characterization of plain textures in terms of posets given by Mustafa Demirci (M. Demirci, Textures and C-spaces, Fuzzy Sets and Systems 158 (11) (2007) 1237–1245), the authors consider the important class of plain ditopological texture spaces and give several new results.

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1. Introduction

An adequate introduction to the theory of ditopological texture spaces may be obtained from [3–7], and many of the major definitions appear also in several recent papers such as [2,13,17–19,22–24]. Here we recall only that a *texturing* on a set S is a subset $\mathcal{S}$ of $\mathcal{P}(S)$ that is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(S,\mathcal{S})$ is then called a *texture*. A texture is regarded as a framework in which to do mathematics.

Many of the definitions relating to textures are conveniently expressed in terms of the p-sets $P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$ and the q-sets $Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$, $s \in S$. However, as noted in [2] we may associate with $(S,\mathcal{S})$ the C-space [10] $(S,\mathcal{S}^c)$, and then the frequently occurring relationship $P_{s'} \not\subseteq Q_s$, $s, s' \in S$, is equivalent to $s\omega_S s'$, where ω_S is the *interior relation* for $(S,\mathcal{S}^c)$. In this paper we will use whichever notation seems to be the more convenient in each case.

In the context of textures a *ditopology* (τ,κ) consists of generally unrelated families of open and closed sets $\tau \subseteq \mathcal{S}$, $\kappa \subseteq \mathcal{S}$, respectively. Here τ and κ satisfy

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \Longrightarrow G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau$, $i \in I \Longrightarrow \bigvee_i G_i \in \tau$,

and

* Corresponding author.
E-mail addresses: ismail@ibu.edu.tr (İ.U. Tiryaki), brown@hacettepe.edu.tr (L.M. Brown).

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(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \Longrightarrow K_1 \cup K_2 \in \kappa$, and
(3) $K_i \in \kappa$, $i \in I \Longrightarrow \bigcap K_i \in \kappa$,

respectively. In particular, topological and bitopological [14] spaces may be regarded as ditopological texture spaces on the discrete texture $(X, \mathcal{P}(X))$. So also may be topologies on Hutton Algebras, although this aspect will not be considered here.

A *plain* texture $(S, \mathcal{S})$ is one that satisfies any, and hence all, of the following equivalent conditions:

- Join coincides with union in $\mathcal{S}$.
- $\mathcal{S}$ is closed under arbitrary unions.
- $P_s \not\subseteq Q_s$ for all $s \in S$.
- The interior relation ω of $(S, \mathcal{S})$ is reflexive.

In [9], Mustafa Demirci pointed out that plain textures may be characterized in terms of partially ordered sets (posets). In this paper we take up this topic in greater detail, and present several important new results relating to plain textures and plain ditopological texture spaces. In particular it is shown that:

- The construct of plain textures and ω-preserving mappings in the sense of F. Yıldız and L.M. Brown [22] is a full, isomorphism-closed concretely reflective subconstruct of the construct of textures and ω-preserving mappings.
- Every complementation on a plain texture is grounded in the sense of S. Özçağ and L.M. Brown [19].
- There exists an isomorphism between the construct of weakly pairwise T_0 bitopological spaces and pairwise continuous functions and the construct of plain T_0 ditopological texture spaces and bicontinuous ω-preserving mappings.

The reader may consult [11] for terms from lattice theory not defined here, and our general reference for category theory is [1].

2. Plain textures and posets

We begin by recalling the result of Demirci mentioned in the introduction. It is established in [9] that if $(N, \leqslant)$ is a poset then the set

$$\mathcal{L}_N = \{L \subseteq N \mid n \in L,\ m \leqslant n \Longrightarrow m \in L\}$$

of *lower subsets* of N is a plain texturing of N. Conversely, if $\mathcal{S}$ is a plain texturing of S and we set $s \leqslant s' \Longleftrightarrow P_s \subseteq P_{s'}$ in S then $(S, \leqslant)$ is a poset and $\mathcal{S} = \mathcal{L}_S$.

Lemma 2.1. *If $(N, \leqslant)$ is a poset and $(N, \mathcal{L}_N)$ the corresponding plain texture then for $n \in N$,*

$$P_n = \{m \in N \mid m \leqslant n\}, \qquad Q_n = \{m \in N \mid n \nleqslant m\}.$$

Proof. Straightforward from the definitions. □

In terms of the interior relation ω of $\mathcal{L}_N$ [2] this gives

$$m\omega n \iff P_n \not\subseteq Q_m \iff P_m \subseteq P_n \iff m \in P_n \iff m \leqslant n,$$

so ω coincides with $\leqslant$. As pointed out in [9], the corresponding C-space $(N, \mathcal{L}_N^c)$ is in fact an Alexandroff discrete [15], Alexandroff [16] or A-space [10].

Examples 2.2. (1) If X is a set and $\leqslant$ the discrete ordering on X, that is $x_1 \leqslant x_2 \Longleftrightarrow x_1 = x_2$, then every subset of X is a lower set and we have $(X, \mathcal{L}_X) = (X, \mathcal{P}(X))$, the discrete texture on X.

(2) Consider the set $\mathbb{I} = [0, 1]$ under the usual ordering $\leqslant$. Clearly the lower sets are $\emptyset$, $\mathbb{I}$ and all sets of the form $[0, r)$, $[0, r]$ for $0 < r < 1$. Hence in this case $\mathcal{L}_{\mathbb{I}} = \mathcal{I}$ and so $(\mathbb{I}, \mathcal{L}_{\mathbb{I}}) = (\mathbb{I}, \mathcal{I})$, the unit interval texture.

(3) Consider the set $\mathbb{R}$ of real numbers under the usual ordering $\leqslant$. Clearly the lower sets are $\emptyset$, $\mathbb{R}$ and all sets of the form $(-\infty, r)$, $(-\infty, r]$ for $r \in \mathbb{R}$. Hence in this case $\mathcal{L}_{\mathbb{R}} = \mathcal{R}$ and so $(\mathbb{R}, \mathcal{L}_{\mathbb{R}}) = (\mathbb{R}, \mathcal{R})$, the real texture.

We will denote by **ifTex** the construct of textures and ω-preserving mappings between them [21], where $\varphi : (S, \mathcal{S}) \to (T, \mathcal{T})$ is ω-preserving if $s\omega_S s' \Longrightarrow \varphi(s)\omega_T\varphi(s')$ for all $s, s' \in S$. Certain subconstructs of **ifTex** were considered in [22,23]. The full subconstruct whose objects are plain textures is denoted by **ifPTex**. Clearly, **ifPTex** coincides with **fPTex** because point functions between plain textures always satisfy the additional condition (b) of [5, Proposition 3.7]. As seen in [5], **fPTex** is isomorphic with **dfPTex**, an isomorphism being given by the identity on objects and the mapping $\varphi \mapsto (f_\varphi, F_\varphi)$

between morphisms. Here we recall that if $\varphi : (S, \mathcal{S}) \to (T, \mathcal{T})$ is ω-preserving — equivalently satisfies the condition (a) of [5, Proposition 3.7] — then (f_φ, F_φ) is the difunction given by

$$f_\varphi = \bigvee\{\overline{P}_{(s,t)} \mid \exists u \in S \text{ satisfying } P_s \not\subseteq Q_u \text{ and } P_{\varphi(u)} \not\subseteq Q_t\},$$
$$F_\varphi = \bigcap\{\overline{Q}_{(s,t)} \mid \exists u \in S \text{ satisfying } P_u \not\subseteq Q_s \text{ and } P_t \not\subseteq Q_{\varphi(u)}\}.$$

Moreover, for $B \in \mathcal{T}$ we have $f_\varphi^{\leftarrow} B = \varphi^{\leftarrow} B = F_\varphi^{\leftarrow} B$, where

$$\varphi^{\leftarrow} B = \bigvee\{P_u \mid \varphi(u) \in B\} = \bigcap\{Q_v \mid \varphi(v) \notin B\}. \tag{2.1}$$

In case $(S, \mathcal{S})$ is plain, so that φ satisfies (b) as well as (a), we see by [5, Lemma 3.9] that $\varphi^{\leftarrow} B = \varphi^{-1}[B]$ for all $B \in \mathcal{T}$. In this case $\varphi^{-1}[B] \in \mathcal{S}$ for all $B \in \mathcal{T}$, so φ is an affine transformation in the sense of [12], or equivalently a textural morphism in the sense of [9]. On the other hand, suppose that φ is an affine transformation and that $(T, \mathcal{T})$ is plain. Take $s, s' \in S$ with $P_s \not\subseteq Q_{s'}$. Then $s \in \varphi^{-1}[P_{\varphi(s)}] \in \mathcal{S}$ by hypothesis as $P_{\varphi(s)} \in \mathcal{T}$, so $s' \in P_s \subseteq \varphi^{-1}[P_{\varphi(s)}]$ and hence $\varphi(s') \in P_{\varphi(s)}$. Since $(T, \mathcal{T})$ is plain this is equivalent to $P_{\varphi(s)} \not\subseteq Q_{\varphi(s')}$ and we have established that φ satisfies (a). Hence, if both $(S, \mathcal{S})$ and $(T, \mathcal{T})$ are plain, the affine transformations coincide with the ω-preserving mappings. This means that as far as plain textures are concerned the choice between the various forms of morphism is a matter of convenience. In the case of general textures, however, these all produce quite different categories. In this paper we will find it convenient to consider **ifTex**, that is consider as morphisms point functions that are required only to be ω-preserving.

The image and co-image operators for **ifTex**-morphisms are also available, if required. Of course, in general, not every difunction can be represented by an **ifTex**-morphism. We note the following for future reference.

Lemma 2.3. *Let $(S_k, \mathcal{S}_k)$, $k = 1, 2, 3$, be textures, $S_1 \xrightarrow{\psi} S_2 \xrightarrow{\varphi} S_3$, ω-preserving mappings and $A \in \mathcal{S}_3$. Then $(\varphi \circ \psi)^{\leftarrow} A = \psi^{\leftarrow}(\varphi^{\leftarrow} A)$.*

Proof. Since $\varphi^{-1}[A] \subseteq \varphi^{\leftarrow} A$ it is easy to deduce $(\varphi \circ \psi)^{\leftarrow} A \subseteq \psi^{\leftarrow}(\varphi^{\leftarrow} A)$. Suppose the reverse inclusion does not hold. Then we have $s \in S_1$ with $\psi^{\leftarrow}(\varphi^{\leftarrow} A) \not\subseteq Q_s$ and $P_s \not\subseteq (\varphi \circ \psi)^{\leftarrow} A$. Take $u \in S_1$ with $\psi^{\leftarrow}(\varphi^{\leftarrow} A) \not\subseteq Q_u$ and $P_u \not\subseteq Q_s$. Then $\psi(u) \in \varphi^{\leftarrow} A$ by (2.1), $P_{\psi(u)} \not\subseteq Q_{\psi(s)}$ as ψ is ω-preserving, so $\varphi^{\leftarrow} A \not\subseteq Q_{\psi(s)}$ and hence $\varphi(\psi(s)) \in A$ by (2.1). On the other hand $(\varphi \circ \psi)(s) \notin A$ by (2.1), which is a contradiction. □

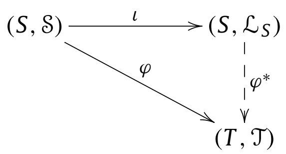
Proposition 2.4. **ifPTex** *is a full, isomorphism-closed concretely reflective subconstruct of* **ifTex**.

Proof. Fullness is clear. To show **ifPTex** is isomorphism-closed, let $(S, \mathcal{S})$ be plain and $(T, \mathcal{T})$ a texture for which there is an **ifTex**-isomorphism $\varphi : S \to T$. By hypothesis ω_S is reflexive, so $\varphi(s)\omega_T\varphi(s)$ for all $s \in S$ as φ is ω-preserving. Clearly φ is onto, so ω_T is reflexive and hence $(T, \mathcal{T})$ is plain.

Now take $(S, \mathcal{S}) \in \mathrm{Ob}\,$**ifTex** and define the reflexive relation $\leqslant$ on S by

$$s_1 \leqslant s_2 \iff s_1 = s_2 \quad \text{or} \quad s_1 \omega_S s_2.$$

Clearly $(S, \leqslant)$ is a poset, and we consider $(S, \mathcal{L}_S) \in \mathrm{Ob}\,$**ifPTex**. The identity mapping $\iota : S \to S$ may be regarded as an **ifTex**-morphism $(S, \mathcal{S}) \to (S, \mathcal{L}_S)$ since $s_1 \omega_S s_2 \Longrightarrow s_1 \leqslant s_2$. We show that it is an **ifPTex**-reflection arrow. Take $(T, \mathcal{T}) \in \mathrm{Ob}\,$**ifPTex** and an **ifTex**-morphism $\varphi : (S, \mathcal{S}) \to (T, \mathcal{T})$.



The only mapping $\varphi^* : S \to T$ making the above diagram commutative is $\varphi^* = \varphi$, so it remains to show that φ is an **ifPTex** morphism from $(S, \mathcal{L}_S)$ to $(T, \mathcal{T})$. Take $s_1, s_2 \in S$ with $s_1 \leqslant s_2$. If $s_1 = s_2$ then $\varphi(s_1)\omega_T\varphi(s_2)$ since ω_T is reflexive. On the other hand if $s_1\omega_S s_2$ then $\varphi(s_1)\omega_T\varphi(s_2)$ since $\varphi : (S, \mathcal{S}) \to (T, \mathcal{T})$ is an **ifTex**-morphism. This completes the proof that $\iota : (S, \mathcal{S}) \to (S, \mathcal{L}_S)$ is an **ifPTex**-reflection arrow.

The fact that **ifPTex** is concretely reflective in **ifTex** is immediate since the **ifPTex**-reflection arrow ι is identity-carried. □

We recall the construct **fTex** from [5, p. 194]. We note that ι in the above proposition need not be an **fTex**-morphism. To see this, let $(S, \mathcal{S})$ be the texture $S = (0, 1]$, $\mathcal{S} = \{(0, r] \mid r \in (0, 1]\} \cup \{\emptyset\}$. Here $r_1 \omega_S r_2 \Longleftrightarrow r_1 < r_2$, so the relation $\leqslant$ on S is the usual partial order on $(0, 1]$ and we have $\mathcal{L}_S = \{(0, r] \mid r \in (0, 1]\} \cup \{(0, r) \mid r \in (0, 1]\} \cup \{\emptyset\}$. Since $\iota^{-1}(0, r) = (0, r) \notin \mathcal{S}$ we see that $\iota : (S, \mathcal{S}) \to (S, \mathcal{L}_S)$ is not an **fTex**-morphism. This is one of the main reasons for considering **ifTex** in place of **fTex**.

When considering morphisms between plain textures, we may use the representation in terms of posets to give explicit formulae for the various terms considered above. If $(N,\leqslant)$, $(M,\leqslant)$ are posets and $\varphi : N \to M$ an order preserving point function then φ is also ω-preserving and so we have a difunction $(f_\varphi, F_\varphi) = (f, F) : (N, \mathcal{L}_N) \to (M, \mathcal{L}_M)$ as described above.

Lemma 2.5. *Let $\varphi : N \to M$ be an order preserving point function between the posets $(N,\leqslant)$ and $(M,\leqslant)$. Then*:

(1) *(f_φ, F_φ) is given explicitly by*:
  (i) $f_\varphi = \{(n,m) \mid n \in N,\ m \in M,\ m \leqslant \varphi(n)\}$,
  (ii) $F_\varphi = \{(n,m) \mid n \in N,\ m \in M,\ \varphi(n) \nleqslant m\}$.
(2) *For $B \in \mathcal{L}_M$ we have $f_\varphi^{\leftarrow} B = \varphi^{-1}[B] = F_\varphi^{\leftarrow} B$.*
(3) *For $A \in \mathcal{L}_N$ we have*:
  (i) $f_\varphi^{\rightarrow} A = \{m \in M \mid \exists n \in A \text{ with } m \leqslant \varphi(n)\}$,
  (ii) $F_\varphi^{\rightarrow} A = \{m \in M \mid \forall n \in N,\ \varphi(n) \leqslant m \Longrightarrow n \in A\}$.

Proof. (1) These equalities follow directly from the general formulae given in [5, Lemma 3.4], and we omit a detailed proof.

(2) Since the textures under consideration are plain, this is just [5, Lemma 3.9].

(3) We verify (i), leaving the dual proof of (ii) to the interested reader. From [5, Definition 2.5(1)] we have

$$f_\varphi^{\rightarrow} A = \bigcap \{Q_t \mid t \in M,\ \forall s \in N,\ f_\varphi \not\subseteq \overline{Q}_{(s,t)} \Longrightarrow A \subseteq Q_s\}.$$

Noting that $f_\varphi \not\subseteq \overline{Q}_{(s,t)}$ is equivalent to $t \leqslant \varphi(s)$ by (1i) we obtain

$$f_\varphi^{\rightarrow} A = \bigcap \bigl\{Q_t \bigm| s \in A \Longrightarrow t \nleqslant \varphi(s)\bigr\}. \tag{2.2}$$

Now take $m \in M$ with $m \leqslant \varphi(n)$ for some $n \in A$, and take $t \in M$ satisfying $s \in A \Longrightarrow t \nleqslant \varphi(s)$. Then $t \nleqslant \varphi(n)$ and so $t \nleqslant m$, that is $m \in Q_t$ by Lemma 2.1 and so $m \in f_\varphi^{\rightarrow} A$.

Conversely, take $m \in f_\varphi^{\rightarrow} A$ and assume that $m \nleqslant \varphi(n)$ for all $n \in A$. Then by (2.2) we have $f_\varphi^{\rightarrow} A \subseteq Q_m$, which gives the contradiction $m \in Q_m$, i.e. $m \nleqslant m$. This establishes the stated equality for $f_\varphi^{\rightarrow} A$. □

Corollary 2.6. *With the notation as above, the difunction (f_φ, F_φ) is surjective if and only if φ is onto, and injective if and only if $\varphi(n_1) \leqslant \varphi(n_2) \Longrightarrow n_1 \leqslant n_2$ for all $n_1, n_2 \in N$. In particular, if (f_φ, F_φ) is injective then φ is one-to-one. Conversely, if φ is one-to-one and onto then (f_φ, F_φ) is injective if and only if φ^{-1} is ω-preserving.*

Proof. These results follow easily from [5, Definition 2.30] and Lemma 2.5(1). □

We will continue to write $\varphi^{\leftarrow} B = \varphi^{-1}[B]$ for the inverse image, $\varphi^{\rightharpoonup} A = f_\varphi^{\rightarrow} A$, $\varphi^{\rightharpoondown} A = F_\varphi^{\rightarrow} A$ for the image and co-image, respectively, when we wish to associate these explicitly with φ. We recall the following inclusions from [5, Theorem 2.24]:

$$\varphi^{\leftarrow}\bigl(\varphi^{\rightharpoondown} A\bigr) \subseteq A \subseteq \varphi^{\leftarrow}\bigl(\varphi^{\rightharpoonup} A\bigr), \quad \forall A \in \mathcal{L}_N,$$
$$\varphi^{\rightharpoonup}\bigl(\varphi^{\leftarrow} B\bigr) \subseteq B \subseteq \varphi^{\rightharpoondown}\bigl(\varphi^{\leftarrow} B\bigr), \quad \forall B \in \mathcal{L}_m.$$

We also recall from [5] that $\varphi^{\leftarrow}$ preserves arbitrary unions and intersections. The following result is of interest.

Proposition 2.7. *Let N and M be posets.*

(1) *If $\varphi : N \to M$ is order preserving then $\varphi^{\rightharpoonup} : \mathcal{L}_N \to \mathcal{L}_M$ preserves p-sets and arbitrary unions. Conversely, if $\theta : \mathcal{L}_N \to \mathcal{L}_M$ preserves p-sets and arbitrary unions then $\theta = \varphi^{\rightharpoonup}$ for some order preserving mapping $\varphi : N \to M$.*
(2) *If $\varphi : N \to M$ is order preserving then $\varphi^{\rightharpoondown} : \mathcal{L}_N \to \mathcal{L}_M$ preserves q-sets and arbitrary intersections. Conversely, if $\theta : \mathcal{L}_N \to \mathcal{L}_M$ preserves q-sets and arbitrary intersections then $\theta = \varphi^{\rightharpoondown}$ for some order preserving mapping $\varphi : N \to M$.*

Proof. (1) We are first required to prove that $\varphi^{\rightharpoonup} P_n = P_{\varphi(n)}$ for all $n \in N$. For $m \in \varphi^{\rightharpoonup} P_n$ there exists $k \in P_n$ with $m \leqslant \varphi(k)$ by Lemma 2.5(3i). However, since $k \leqslant n$ and φ is order preserving we deduce $m \leqslant \varphi(n)$, whence $m \in P_{\varphi(n)}$. On the other hand, for $m \in P_{\varphi(n)}$ we have $m \leqslant \varphi(n)$, so $m \in \varphi^{\rightharpoonup} P_n$ by Lemma 2.5(3i) since $n \in P_n$. Hence $\varphi^{\rightharpoonup}$ preserves the p-sets, and it preserves arbitrary unions by [5, Corollary 2.12(2)].

Conversely, let $\theta : \mathcal{L}_N \to \mathcal{L}_M$ have the stated properties. For $n \in N$ the set $\theta(P_n)$ is a p-set in $\mathcal{L}_M$, so we have a unique element of M, that we denote by $\varphi(n)$, satisfying $\theta(P_n) = P_{\varphi(n)}$. We must show that the mapping $\varphi : N \to M$ so defined is order preserving. However, if $n \leqslant k$ in N then $P_n \subseteq P_k$, and since θ preserves unions it certainly preserves containment, so $P_{\varphi(n)} = \theta(P_n) \subseteq \theta(P_k) = P_{\varphi(k)}$ and we obtain $\varphi(n) \leqslant \varphi(k)$, as required. It remains to show that $\theta = \varphi^{\rightharpoonup}$. However, for $A \in \mathcal{L}_N$ we have $A = \bigcup_{n \in A} P_n$ and so using the stated properties of θ and the corresponding properties of $\varphi^{\rightharpoonup}$ we have

$$\theta(A) = \bigcup_{n\in A}\theta(P_n) = \bigcup_{n\in A}P_{\varphi(n)} = \bigcup_{n\in A}\varphi^{\to}P_n = \varphi^{\to}\left(\bigcup_{n\in A}P_n\right) = \varphi^{\to}A$$

as required.

(2) The proof is dual to (1), and is omitted. □

The above results are particular to plain textures. In the case of an ω-preserving mapping $\varphi : S \to T$ between general textures it is easy to see that $f_\varphi^{\to} : \mathcal{S} \to \mathcal{T}$ preserves the p-sets if and only if φ satisfies condition (b) of [5, Proposition 3.6], while $F_\varphi^{\to} : \mathcal{S} \to \mathcal{T}$ preserves the q-sets if and only if φ satisfies condition (c) of [6, Lemma 3.8]. Conversely, if $\theta : \mathcal{S} \to \mathcal{T}$ satisfies either of the conditions of Proposition 2.7(1) or (2) then the corresponding mapping φ need not be ω-preserving. Hence if such θ are used as morphisms the resulting categories will be isomorphic to **ifPTex** in the plain case, but will produce different categories in general.

We now consider complementations [18, p. 3296] on plain textures. Hence a complementation on $(N, \mathcal{L}_N)$ is an inclusion-reversing involution $\sigma : \mathcal{L}_N \to \mathcal{L}_N$, and we begin with a useful characterization.

Proposition 2.8. *Let σ be a complementation on $(N, \mathcal{L}_N)$ and define $\underline{\sigma} : N \to \mathcal{L}_N$ by $\underline{\sigma}(n) = \sigma(P_n)$ for all $n \in N$. Then:*

(i) $\forall m, n \in N,\ n \leqslant m \Longleftrightarrow \underline{\sigma}(m) \subseteq \underline{\sigma}(n)$.
(ii) $\forall m, n \in N,\ m \in \underline{\sigma}(n) \Longrightarrow n \in \underline{\sigma}(m)$.
(iii) $\forall n \in N,\ A \in \mathcal{L}_N,\ n \notin A \Longrightarrow \exists m \in N$ *with* $A \subseteq \underline{\sigma}(m)$ *and* $n \notin \underline{\sigma}(m)$.

Conversely, if $\underline{\sigma} : N \to \mathcal{L}_N$ is a mapping satisfying the conditions (i)–(iii) *above, then $\sigma : \mathcal{L}_N \to \mathcal{L}_N$ defined by*

$$\sigma(A) = \bigcap\{\underline{\sigma}(m) \mid m \in A\} \tag{2.3}$$

is a complementation on $\mathcal{L}_N$ satisfying $\underline{\sigma}(n) = \sigma(P_n)$ for each $n \in N$.

Proof. (i) $n \leqslant m \Longleftrightarrow P_n \subseteq P_m \Longleftrightarrow \sigma(P_m) \subseteq \sigma(P_n) \Longleftrightarrow \underline{\sigma}(m) \subseteq \underline{\sigma}(n)$.

(ii) $m \in \underline{\sigma}(n) \Longrightarrow P_m \subseteq \sigma(P_n) \Longrightarrow P_n = \sigma(\sigma(P_n)) \subseteq \sigma(P_m) \Longrightarrow n \in \underline{\sigma}(m)$.

(iii) If $n \notin A$ then $P_n \not\subseteq A$ so $\sigma(A) \not\subseteq \sigma(P_n) = \underline{\sigma}(n)$. Taking $m \in \sigma(A)$ with $m \notin \underline{\sigma}(n)$ gives $A \subseteq \underline{\sigma}(m)$ and $n \notin \underline{\sigma}(m)$ by (ii).

Conversely, let $\underline{\sigma} : N \to \mathcal{L}_N$ be given satisfying (i)–(ii), and for $A \in \mathcal{L}_N$ define $\sigma(A)$ by (2.3). Certainly $\sigma(A) \in \mathcal{L}_N$, so $\sigma : \mathcal{L}_N \to \mathcal{L}_n$ is a mapping.

For $A \subseteq B$ in $\mathcal{L}_N$ it is clear that $\sigma(B) \subseteq \sigma(A)$, so to show that σ is a complementation it remains to show that $\sigma(\sigma(A)) = A$ for any $A \in \mathcal{L}_N$. We begin by showing that

$$P_m = \sigma\big(\underline{\sigma}(m)\big), \quad \forall m \in N. \tag{2.4}$$

By definition $\sigma(\underline{\sigma}(m)) = \bigcap\{\underline{\sigma}(k) \mid k \in \underline{\sigma}(m)\}$, and $k \in \underline{\sigma}(m) \Longrightarrow m \in \underline{\sigma}(k) \Longrightarrow P_m \subseteq \underline{\sigma}(k)$ by (ii), so certainly $P_m \subseteq \sigma(\underline{\sigma}(m))$. On the other hand $n \notin P_m \Longrightarrow n \nleqslant m \Longrightarrow \underline{\sigma}(m) \not\subseteq \underline{\sigma}(n)$ by (i), so we may take $k \in \underline{\sigma}(m)$ satisfying $k \notin \underline{\sigma}(n)$. By (ii) we have $n \notin \underline{\sigma}(k)$, and so $n \notin \sigma(\underline{\sigma}(m))$ which establishes $\sigma(\underline{\sigma}(m)) \subseteq P_m$ and hence (2.4).

Now suppose that $\sigma(\sigma(A)) \not\subseteq A$ for some $A \in \mathcal{L}_N$ and take $n \in \sigma(\sigma(A))$ with $n \notin A$. By (iii) we have $m \in N$ satisfying $A \subseteq \underline{\sigma}(m)$ and $n \notin \underline{\sigma}(m)$. The first inclusion gives $P_m = \sigma(\underline{\sigma}(m)) \subseteq \sigma(A)$ by (2.4) so $m \in \sigma(A)$. Now using (2.3) with A replaced by $\sigma(A)$ gives $n \in \sigma(\sigma(A)) \subseteq \underline{\sigma}(m)$, which is a contradiction. Hence, $\sigma(\sigma(A)) \subseteq A$.

To prove the opposite inclusion take $n \in \sigma(A)$. Then by (2.3) we have $n \in \underline{\sigma}(m)$, $\forall m \in A$, and so $m \in \underline{\sigma}(n)$, $\forall m \in A$ by (ii). Hence $n \in \sigma(A) \Longrightarrow A \subseteq \underline{\sigma}(n)$, so $A \subseteq \sigma(\sigma(A))$ by (2.3) with A replaced by $\sigma(A)$.

This completes the proof that σ is a complementation, and (2.4) now gives $\sigma(P_m) = \sigma(\sigma(\underline{\sigma}(m))) = \underline{\sigma}(m)$, as required. □

The concept of grounded complementation was introduced in [19, Definition 4.4], and it was pointed out that [6, Example 2.14] describes a texture with a non-grounded complementation.

Definition 2.9. A complementation σ on the texture $(S, \mathcal{S})$ is called *grounded* if there exists an involution $s \mapsto s'$ on S satisfying $\sigma(P_s) = Q_{s'}$ for all $s \in S$.

Theorem 2.10. *Any complementation σ on the plain texture $(N, \mathcal{L}_N)$ is grounded, and the corresponding involution $n \mapsto n'$ is order-reversing. Conversely, if $n \mapsto n'$ is an order-reversing involution on $(N, \leqslant)$ then $\underline{\sigma}(n) = Q_{n'}$ defines a grounded complementation σ on $\mathcal{L}_N$ for which $\underline{\sigma}(n) = \sigma(P_n)$ for all $n \in N$.*

Proof. Let σ be a complementation on $(N, \mathcal{L}_N)$ and take $n \in N$. Since $n \notin Q_n$ we have by Proposition 2.8(iii) applied to $A = Q_n$ an element $n' \in N$ satisfying $Q_n \subseteq \sigma(P_{n'})$ and $n \notin \sigma(P_{n'})$. The latter gives $\sigma(P_{n'}) \subseteq Q_n$ and we have $\sigma(P_{n'}) = Q_n$.

Clearly n' is the only element of N satisfying $\sigma(P_{n'}) = Q_n$, so we have a mapping $n \mapsto n'$ on N. We must show that $(n')' = n$, whence $n \to n'$ is an involution and $\sigma(P_n) = Q_{n'}$, that is, σ is grounded.

Firstly, $n \notin Q_n = \sigma(P_{n'}) \Longrightarrow n' \notin \sigma(P_n)$ by Proposition 2.8(ii) and so $\sigma(P_n) \subseteq Q_{n'} = \sigma(P_{(n')'})$. This gives $P_{(n')'} \subseteq P_n$. Secondly, $n' \notin Q_{n'} = \sigma(P_{(n')'})$ gives $(n')' \notin \sigma(P_{n'}) = Q_n$ and so $n \leqslant (n')'$, that is $P_n \subseteq P_{(n')'}$. These two inclusions show that $P_n = P_{(n')'}$ and hence $n = (n')'$, as required.

Since $\sigma(P_n) = Q_{n'}$ for the involution $n \mapsto n'$ on N we have $\underline{\sigma}(n) = Q_{n'}$, so by condition (i) of Proposition 2.8 we have

$$n \leqslant m \quad \Longrightarrow \quad \underline{\sigma}(m) \subseteq \underline{\sigma}(n) \quad \Longrightarrow \quad Q_{m'} \subseteq Q_{n'} \quad \Longrightarrow \quad m' \leqslant n'$$

by Lemma 2.1. Hence the involution $n \mapsto n'$ is order-reversing.

For the converse let $n \mapsto n'$ be an order-reversing involution on $(N, \leqslant)$. We need only verify conditions (i)–(iii) of Proposition 2.8.

(i) $n \leqslant m \Longrightarrow m' \leqslant n' \Longrightarrow Q_{m'} \subseteq Q_{n'} \Longrightarrow \underline{\sigma}(m) \subseteq \underline{\sigma}(n)$.

(ii) $m \in \underline{\sigma}(n) = Q_{n'} \Longrightarrow n' \nleqslant m \Longrightarrow m' \nleqslant n \Longrightarrow n \in Q_{m'} = \underline{\sigma}(m)$.

(iii) For $A \in \mathcal{L}_N$ we have $A = \bigcap\{Q_k \mid k \notin A\}$ by [5, Theorem 1.2(6)]. Hence if $n \notin A$ we have $k \in N$ with $n \notin Q_k$ and $k \notin A$. Setting $m = k'$ now gives $A \subseteq \underline{\sigma}(m)$ and $n \notin \underline{\sigma}(m)$. □

Examples 2.11. (1) Consider a set X with the discrete order. Then $\mathcal{L}_X = \mathcal{P}(X)$ and any involution $x \mapsto x'$ on X is order-reversing, so defines a complementation on $\mathcal{P}(X)$. Let us identify the complementation σ corresponding to the identity involution $x \mapsto x$, i.e. $x' = x$, $\forall x \in X$. For $x \in X$ we have $\sigma(\{x\}) = \sigma(P_x) = Q_x = X \setminus \{x\}$. Hence, for $A \in \mathcal{P}(X)$ we have

$$\sigma(A) = \sigma\left(\bigcup_{x\in A}\{x\}\right) = \bigcap_{x\in A}\sigma\bigl(\{x\}\bigr) = \bigcap_{x\in A}\bigl(X \setminus \{x\}\bigr) = X \setminus A.$$

This shows that $\sigma = \pi_X$, the set-theoretic complement on X.

(2) For the unit interval $(\mathbb{I}, \leqslant)$ with its usual order, we have already seen that $\mathcal{L}_{\mathbb{I}} = \mathcal{I}$. Clearly $r \mapsto 1 - r$, $r \in \mathbb{I}$, is an order reversing involution on $\mathbb{I}$ and so defines a complementation on $\mathcal{I}$. It is straightforward to verify that this is the standard complementation ι given by $\iota([0, r]) = [0, 1 - r)$ and $\iota([0, r)) = [0, 1 - r]$, $r \in \mathbb{I}$.

(3) For the set $(\mathbb{R}, \leqslant)$ of real numbers with the usual ordering, the involution $r \mapsto -r$ is clearly order reversing. It defines a complementation σ on $\mathcal{L}_{\mathbb{R}} = \mathcal{R}$ given by $\sigma((-\infty, r]) = (-\infty, -r)$ and $\sigma((-\infty, r)) = (-\infty, -r]$, $r \in \mathbb{R}$.

Proposition 2.12. *Let $(N, \leqslant)$, $(M, \leqslant)$ be posets and let complementations σ, θ on $(N, \mathcal{L}_N)$, $(M, \mathcal{L}_M)$ be given by order reversing involutions $n \mapsto n'$, $m \mapsto m'$, respectively. If $\varphi : N \to M$ is order preserving then the complement $(f_\varphi, F_\varphi)' = ((F_\varphi)', (f_\varphi)')$ of the difunction $(f_\varphi, F_\varphi) : (N, \mathcal{L}_N, \sigma) \to (M, \mathcal{L}_M, \theta)$ is given by*

$$(f_\varphi)' = \bigl\{(n, m) \bigm| n \in N,\ m \in M,\ m' \nleqslant \varphi\bigl(n'\bigr)\bigr\},$$
$$(F_\varphi)' = \bigl\{(n, m) \bigm| n \in N,\ m \in M,\ \varphi\bigl(n'\bigr) \leqslant m'\bigr\}.$$

Proof. We establish the formula for $(f_\varphi)'$, leaving the dual proof of the formula for $(F_\varphi)'$ to the reader.

We recall the formula

$$(f_\varphi)' = \bigcap\bigl\{\overline{Q}_{(s,t)} \bigm| \exists u, v \text{ with } f_\varphi \nsubseteq \overline{Q}_{(u,v)},\ \sigma(Q_s) \nsubseteq Q_u,\ P_v \nsubseteq \theta(P_t)\bigr\}$$

from [5, Definition 2.18(1)]. Since we are assuming w.l.o.g. by Theorem 2.10 that the complementations are grounded we have $\sigma(Q_s) = P_{s'}$, $\theta(P_t) = Q_{t'}$ and we easily obtain:

$$(f_\varphi)' = \bigcap\bigl\{\overline{Q}_{(s,t)} \bigm| t' \leqslant \varphi\bigl(s'\bigr)\bigr\}. \tag{2.5}$$

Now take $n \in N$, $m \in M$ with $m' \nleqslant \varphi(n')$ and $s \in N, t \in M$ with $t' \leqslant \varphi(s')$. We must show $(n, m) \in \overline{Q}_{(s,t)}$. Since $\overline{Q}_{(s,t)} = ((N \setminus \{s\}) \times M) \cup (N \times Q_t)$ this is clear for $s \neq n$, so we assume $s = n$. Now $t' \leqslant \varphi(n') \Longrightarrow m' \nleqslant t' \Longrightarrow t \nleqslant m \Longrightarrow m \in Q_t$, and again we have $(n, m) \in \overline{Q}_{(s,t)}$. Hence $(n, m) \in (f_\varphi)'$.

Conversely, take $(n, m) \in (f_\varphi)'$ and suppose that $m' \leqslant \varphi(n')$. Then $(f_\varphi)' \subseteq \overline{Q}_{(n,m)}$, and we obtain the contradiction $m \in Q_m$. Hence $m' \nleqslant \varphi(n')$, and the stated equality is proved. □

Let us define the point function $\varphi' : N \to M$ by $\varphi'(n) = (\varphi(n'))'$. Since φ is order preserving and the involutions $n \mapsto n'$, $m \mapsto m'$ order reversing it is clear that φ' is order preserving. Moreover:

Corollary 2.13. *With the notation as in Proposition 2.12 we have $(f_\varphi, F_\varphi)' = (f_{\varphi'}, F_{\varphi'})$.*

Proof. Since $m' \nleqslant \varphi(n') \iff (\varphi(n'))' \nleqslant m \iff \varphi'(n) \nleqslant m$ we have $(f_\varphi)' = F_{\varphi'}$ by Proposition 2.12 and Lemma 2.5(1). Likewise, $(F_\varphi)' = f_{\varphi'}$. □

In view of the above result it is natural to call φ' the *complement* of φ. Likewise, φ may be called *complemented* if $\varphi' = \varphi$, since this corresponds to $(f_\varphi, F_\varphi) = (f_\varphi, F_\varphi)'$. Clearly, φ is complemented if and only if it preserves the order-reversing involutions $n \mapsto n'$, $m \mapsto m'$ in the sense that $\varphi(n') = \varphi(n)'$ for all $n \in N$.

We conclude this section by considering the construct **gcifTex** whose objects are textures with a grounded complementation, and whose morphisms are complemented ω-preserving functions between the base sets. By Theorem 2.10 the subconstruct **gcifPTex** may be denoted by **cifPTex**, and we have the following result:

Proposition 2.14. **cifPtex** *is a full, isomorphism-closed concretely reflective subconstruct of* **gcifTex***.*

Proof. Let $(S, \mathcal{S})$ be a texture with grounded complementation σ given by the involution $s \to s'$ on S. We consider the partial order $\leqslant$ on S as in the proof of Proposition 2.4, and the corresponding plain texture $(S, \mathcal{L}_S)$. The involution $s \to s'$ on S is order-reversing for the partial order $\leqslant$. Indeed,

$$s_1 \omega_S s_2 \quad\Longrightarrow\quad P_{s_2} \not\subseteq Q_{s_1} \quad\Longrightarrow\quad P_{s_1'} = \sigma(Q_{s_1}) \not\subseteq \sigma(P_{s_2}) = Q_{s_2'} \quad\Longrightarrow\quad s_2' \omega_S s_1',$$

while $s_1 = s_2 \Longrightarrow s_2' = s_1'$. Hence, by Theorem 2.10 the involution $s \to s'$ defines a complementation σ_S on $(S, \mathcal{L}_S)$, and it is immediate that the morphism ι is complemented. Since the complementations σ and σ_S are both given by the same involution on S, and since the underlying function of φ^* is φ we see that if φ is complemented then so is φ^*. The remainder of the proof is essentially as for Proposition 2.4. □

3. Ditopologies on plain textures

In this section we consider textures in the presence of a ditopology.

An **ifTex** morphism φ between ditopological textures spaces $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is called *continuous* (*cocontinuous*) if $A \in \tau_2 \Longrightarrow \varphi^{\leftarrow} A \in \tau_1$ (resp., $A \in \kappa_2 \Longrightarrow \varphi^{\leftarrow} A \in \kappa_1$). Here $\varphi^{\leftarrow} A$ is given by (2.1). When φ is both continuous and cocontinuous it is referred to as *bicontinuous*. The category of ditopological texture spaces and bicontinuous **ifTex**-morphisms is denoted by **ifDitop** [21]. Clearly we may regard **ifDitop** as a concrete category over **ifTex**, and the forgetful functor is easily seen to be topological [1,8].

The restriction **ifPDitop** of **ifDitop** to plain textures is isomorphic to **fPDitop**, and hence to **dfPDitop**.

Proposition 3.1. **ifPDitop** *is a full, isomorphism-closed concretely reflective subcategory of* **ifDitop***.*

Proof. The fact that **ifPDitop** is full and isomorphism closed in **ifDitop** follows trivially from Proposition 2.4. For $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,$**ifDitop**, define $(S, \mathcal{L}_S)$ and the **ifTex**-morphism $\iota : (S, \mathcal{S}) \to (S, \mathcal{L}_S)$ as in the proof of Proposition 2.4. Let

$$\tau_S = \{A \in \mathcal{L}_S \mid \iota^{\leftarrow} A \in \tau\}, \qquad \kappa_S = \{A \in \mathcal{L}_S \mid \iota^{\leftarrow} A \in \kappa\}.$$

Since the inverse image operator preserves arbitrary joins and intersections, (τ_S, κ_S) is a ditopology on $(S, \mathcal{L}_S)$. We must show that $\iota : (S, \mathcal{S}, \tau, \kappa) \to (S, \mathcal{L}_S, \tau_S, \kappa_S)$ is an **ifPDitop**-reflection arrow. We know it is an **ifPTex**-reflection arrow, so referring to the diagram in the proof of Proposition 2.4 we see that it suffices to show that $\varphi^* = \varphi$ is (τ_S, κ_S)–(τ_T, κ_T) bicontinuous, where (τ_T, κ_T) is a ditopology on $(T, \mathcal{T})$ for which $\varphi : (S, \mathcal{S}, \tau, \kappa) \to (T, \mathcal{T}, \tau_T, \kappa_T)$ is bicontinuous. However, for $B \in \mathcal{T}$ we have $\varphi^{\leftarrow} B = \iota^{\leftarrow}((\varphi^*)^{\leftarrow} B)$ by Lemma 2.3 since $\varphi = \varphi^* \circ \iota$. For $B \in \tau_T$ we have $\varphi^{\leftarrow} B \in \tau$, whence $(\varphi^*)^{\leftarrow} B \in \tau_S$ by definition. Hence, φ^* is continuous, and a similar argument shows that it is cocontinuous. This establishes that ι is an **ifPDitop**-reflection arrow, and it is clearly concrete over **ifTex**. □

Corollary 3.2. **cifPDitop** *is a full, isomorphism-closed concretely reflective subcategory of* **gcifDitop***.*

Proof. Combining the necessary elements of the proofs of Propositions 2.14 and 3.1 we see that it will be sufficient to prove that when (τ, κ) is complemented then (τ_S, κ_S) is complemented too. We note that for $A \in \mathcal{L}_S$ we have $\sigma(\iota^{\leftarrow} A) = \iota^{\leftarrow}(\sigma_S(A))$. Indeed, this may be proved directly, or by applying [5, Lemma 2.20] to the corresponding difunctions and recalling that ι is complemented. Thus,

$$\sigma(\iota^{\leftarrow} A) = \sigma(f_\iota^{\leftarrow} A) = \sigma((f_\iota^{\leftarrow} \sigma_S(\sigma_S(A)))) = (f_\iota')^{\leftarrow} \sigma_S(A) = \iota^{\leftarrow} \sigma_S(A),$$

as required. Now

$$G \in \tau_S \iff \iota^{\leftarrow} G \in \tau \iff \sigma(\iota^{\leftarrow} G) \in \kappa \iff \iota^{\leftarrow} \sigma_S(G) \in \kappa \iff \sigma_S(G) \in \kappa_S,$$

which verifies that (τ_S, κ_S) is complemented when $\sigma(\tau) = \kappa$. □

It is expected that ditopologies on a plain texture will possess various special properties, and we discuss some of these below.

If (τ,κ) is a ditopology on $(N,\mathcal{L}_N)$ then $\tau\subseteq\mathcal{L}_N$ is a topology on N in the usual sense (since join coincides with union for plain textures), that is satisfies the characteristic properties of a family of open sets. Dually, $\kappa\subseteq\mathcal{L}_N$ is a cotopology, that is satisfies the characteristic properties of a family of closed sets. In general there is no *a priori* relation between the families τ and κ. However, if σ is a complementation on $(N,\mathcal{L}_N)$ and we have $G\in\tau\Longleftrightarrow\sigma(G)\in\kappa$ (equivalently, $K\in\kappa\Longleftrightarrow\sigma(K)\in\tau$) then (τ,κ) is called a *complemented ditopology* on $(N,\mathcal{L}_N)$.

Exactly as for classical topology $\beta\subseteq\tau$ is a *base* for τ if every $G\in\tau$ is a union of sets in β, or equivalently if given $n\in G\in\tau$ there exists $B\in\beta$ with $n\in B\subseteq G$. Dually, $\gamma\subseteq\kappa$ is a *cobase* for κ if every $K\in\kappa$ is an intersection of sets in γ, equivalently if given $n\notin K\in\kappa$ there exists $C\in\gamma$ with $n\notin C\supseteq K$.

Again as in the classical case, $\beta\subseteq\mathcal{L}_N$ is a base for some topology on $(N,\mathcal{L}_N)$ if it is a cover of N in the sense that $N\subseteq\bigcup\beta$, and if for $n\in B_1\cap B_2$, $B_1,B_2\in\beta$, there exists $B_3\in\beta$ with $n\in B_3\subseteq B_1\cap B_2$. Hence, any subset of $\mathcal{L}_N$ is a subbase for some topology on $(N,\mathcal{L}_N)$ in the sense that the set of finite intersections of sets in this family has the aforementioned properties of a base. Dually, $\gamma\subseteq\mathcal{L}_N$ is a base for some cotopology on $(N,\mathcal{L}_N)$ if it is a cocover of $\emptyset$ in the sense that $\bigcap\gamma\subseteq\emptyset$, and if given $n\notin C_1\cup C_2$, $C_1,C_2\in\gamma$, there exists $C_3\in\gamma$ with $n\notin C_3\supseteq C_1\cup C_2$. Again, any subset of $\mathcal{L}_N$ is a subbase for some cotopology on $(N,\mathcal{L}_N)$ in the sense that the set of finite unions is a base. These conditions should be compared with the much more complex conditions given for general textures in [6]. In terms of a ditopology (τ,κ), a base (subbase) for τ is known as a base (subbase) of (τ,κ) while a base (subbase) for τ is known as a cobase (cosubbase) of (τ,κ).

There is an even greater simplification in the case of neighborhoods. In $(N,\mathcal{L}_N)$ a nhd of $n\in N$ is a set $A\in\mathcal{L}_N$ for which $n\in G\subseteq A$ for some $G\in\tau$. The set of nhds of n is denoted by $\eta(n)$. Clearly $\eta(n)$ is an $\mathcal{L}_N$-filter (i.e., filter of $\mathcal{L}_N$-sets) with an open base. Neither of these properties need hold in the case of a general texture. Dually, $B\in\mathcal{L}_N$ is a conhd of $n\in N$ if $n\notin K\supseteq B$ for some $K\in\kappa$. The set $\nu(n)$ of conhds of n is an $\mathcal{L}_N$-cofilter with a closed base.

A product $\mathcal{F}\times\mathcal{G}$ of an $\mathcal{L}_N$-filter $\mathcal{F}$ and an $\mathcal{L}_N$-cofilter $\mathcal{G}$ is known as a difilter. The difilter $\mathcal{F}\times\mathcal{G}$ is regular if $\mathcal{F}\cap\mathcal{G}=\emptyset$, which is equivalent to $A\nsubseteq B$ for all $(A,B)\in\mathcal{F}\times\mathcal{G}$. For each $n\in N$, $\eta(n)\times\nu(n)$ is a regular difilter, and we say that the regular difilter $\mathcal{F}\times\mathcal{G}$ diconverges to n if $\eta(n)\times\nu(n)\subseteq\mathcal{F}\times\mathcal{G}$. We also say that n is a dicluster point of $\mathcal{F}\times\mathcal{G}$ if $n\in[A]\setminus]B[$ for all $(A,B)\in\mathcal{F}\times\mathcal{G}$. In parallel with the situation in classical topology, a dilimit point is a dicluster point, while a dicluster point of a maximal regular difilter is a dilimit point. The reader is referred to [20] for a detailed analysis which includes the much more involved non-plain case.

The space $(N,\mathcal{L}_N,\tau,\kappa)$ is called compact (stable) if every τ-open cover of N (of any $K\in\kappa\setminus\{N\}$) has a finite subcover. Dually, it is cocompact (costable) if every κ-closed cocover of $\emptyset$ (of any $G\in\tau\setminus\{\emptyset\}$) has a finite subcocover. A compact, stable, cocompact and costable space is called dicompact. The following result is essentially [20, Theorem 2.20] specialized to the case of plain textures.

Theorem 3.3. *The following are equivalent*:

(1) $(N,\mathcal{L}_N,\tau,\kappa)$ *is dicompact.*
(2) *Every regular difilter on* $(N,\mathcal{L}_N,\tau,\kappa)$ *has a dicluster point* $n\in N$.
(3) *Every maximal regular difilter on* $(N,\mathcal{L}_N,\tau,\kappa)$ *diconverges to some* $n\in N$.

This gives a characterization of dicompactness that echoes a characterization of compactness in topological spaces. To give a corresponding covering property, we recall the notion of dicover from [3]. An indexed family $\mathcal{C}=\{(A_j,B_j)\mid j\in J\}$ is a dicover of $(N,\mathcal{L}_N)$ if $A_j,B_j\in\mathcal{L}_N$, $j\in J$, and for any partition J_1,J_2 of J, $\bigcap_{j\in J_1}B_j\subseteq\bigcup_{j\in J_2}A_j$. We have:

Lemma 3.4. $\mathcal{C}=\{(A_j,B_j)\mid j\in J\}$ *is a dicover on* $(N,\mathcal{L}_N)$ *if and only if given* $n\in N$ *there exists* $j\in J$ *for which* $n\in A_j\setminus B_j$.

Proof. Let $\mathcal{C}$ be a dicover and suppose that for some $n\in N$ we have $n\notin A_j\setminus B_j$ for all $j\in J$. Setting $J_1=\{j\in J\mid n\in B_j\}$, $J_2=J\setminus J_1$ gives $n\in\bigcap_{j\in J_1}B_j\subseteq\bigcup_{j\in J_2}A_j$, which is a contradiction. Conversely, it is easily seen that the stated condition ensures that $\mathcal{C}$ is a dicover. □

The condition in Lemma 3.4 is often easier to check than the original definition. For general textures the proof of necessity given above does not hold since we then have $\bigvee_{j\in J_2}A_j$ in place of $\bigcup_{j\in J_2}A_j$, and indeed this condition need not be satisfied for non-plain textures.

A dicover $\mathcal{C}=\{(A_j,B_j)\mid j\in J\}$ is finite if the set $\{A_j\mid j\in J\}$ is finite, and cofinite if the set $\{B_j\mid j\in J\}$ is finite. Moreover, if $\mathcal{C}$ is defined on a ditopological texture space it is said to be open, coclosed if $A_j\in\tau$, $B_j\in\kappa$ for all $j\in J$. Restating part of [3, Theorem 3.5] for the plain case, and noting that in that paper finite, cofinite was just called finite, we now have:

Theorem 3.5. *The following are equivalent*:

(1) $(N,\mathcal{L}_N,\tau,\kappa)$ *is dicompact.*
(2) *Every open, coclosed dicover on* $(N,\mathcal{L}_N,\tau,\kappa)$ *has a finite, cofinite subdicover.*

A closely integrated system of separation axioms for general ditopological texture spaces is discussed in some detail in [7]. These were derived from the strong separation axioms for bitopological spaces by setting up a functor from the construct **Bitop** of bitopological spaces and pairwise continuous functions to the category **dfDitop**. See [7] for a summary of bitopological separation axioms using the notation of Ralph Kopperman [16]. This functor was obtained by associating with a bitopological space (X,u,v) in the sense of [14] the ditopological texture space $(X,\mathcal{P}(X),u,v^c)$. In particular, a characteristic property of the resulting T_0 axiom for ditopological texture spaces given in [7, Theorem 4.7(4)] leads to the following:

$(N,\mathcal{L}_N,\tau,\kappa)$ is T_0 if for $m \not\leqslant n$ we have $C \in \tau \cup \kappa$ with $n \in C$ and $m \notin C$.

We recall that a bitopological space (X,u,v) is called *strongly pairwise T_0* if

$$x \in \overline{y}^v \text{ and } y \in \bar{x}^u \quad \Longrightarrow \quad x = y.$$

This bitopological T_0 axiom is often too strong to be useful. It has been observed, however, that when applied to many important ditopological texture spaces, such as the unit interval and real spaces, the corresponding ditopological T_0 separation axiom behaves more like the bitopological *weak pairwise T_0 axiom*,

$$x \in \overline{y}^u \cap \overline{y}^v \text{ and } y \in \bar{x}^u \cap \bar{x}^v \quad \Longrightarrow \quad x = y.$$

Why this is so, at least in the case of plain textures, we now see by setting up a new functor. We begin with a useful lemma.

Lemma 3.6.

(1) *Let N be a set and $\gamma \subseteq \mathcal{P}(N)$. Then the relation $\leqslant_\gamma$ defined on N by*

$$m \leqslant_\gamma n \quad \Longleftrightarrow \quad (\forall C \in \gamma) \quad (n \in C \Longrightarrow m \in C)$$

is a partial order if and only if γ separates the points of N. In this case the set $\mathcal{L}_N^\gamma$ of lower sets of $(N,\leqslant_\gamma)$ is the smallest plain texturing of N containing γ.

(2) *If $(N,\leqslant)$ is a poset and $\gamma \subseteq \mathcal{L}_N$ separates the points of N, then $\mathcal{L}_N^\gamma \subseteq \mathcal{L}_N$. We have $\mathcal{L}_N^\gamma = \mathcal{L}_N$ if and only if γ satisfies*

$$m \not\leqslant n \quad \Longrightarrow \quad \exists C \in \gamma \text{ satisfying } n \in C \text{ and } m \notin C.$$

Proof. (1) The first statement is immediate. For the second, $\gamma \subseteq \mathcal{L}_N^\gamma$ is clear. For $m \not\leqslant_\gamma n$ in N, choose $C_m^n \in \gamma$ with $n \in C_m^n$, $m \notin C_m^n$. Then it is easy to verify that $A \in \mathcal{L}_N^\gamma$ may be expressed in the form

$$A = \bigcup \Big\{ \bigcap \{ C_m^n \mid m \not\leqslant_\gamma n \} \mid n \in N \Big\},$$

which completes the proof of (1).

(2) The inclusion follows from (1), or from the evident fact that $\leqslant \subseteq \leqslant_\gamma$. Clearly the given condition is equivalent to $\leqslant_\gamma \subseteq \leqslant$, and hence to $\mathcal{L}_N^\gamma = \mathcal{L}_N$. □

To apply this lemma, let (N,u,v) be a bitopological space and $\gamma = u \cup v^c$. We note at once that γ separates the points of N if and only if (N,u,v) is weakly pairwise T_0. Hence, if (N,u,v) is weakly pairwise T_0, it may be associated with the (plain) ditopological texture space $(N,\mathcal{L}_N^\gamma,u,v^c)$. Moreover, applying Lemma 3.6(2) with $\leqslant = \leqslant_\gamma$, we see that $(N,\mathcal{L}_N^\gamma,u,v^c)$ is T_0. This gives us a mapping $\mathfrak{K}$ from the objects of the construct $\mathbf{Bitop_{w0}}$ of weakly pairwise T_0 bitopological spaces and pairwise continuous functions to the objects of the construct $\mathbf{ifPDitop_0}$ of plain T_0 ditopological texture spaces and ω-preserving bicontinuous functions.

To deal with the morphisms, let $\varphi : (N_1,u_1,v_1) \to (N_2,u_2,v_2)$ be pairwise continuous. Setting $\gamma_k = u_k \cup v_k^c$, $k = 1,2$, we show that $\varphi : (N_1,\leqslant_{\gamma_1}) \to (N_2,\leqslant_{\gamma_2})$ is order preserving. Take $m,n \in N_1$ with $m \leqslant_{\gamma_1} n$, and suppose that $\varphi(m) \not\leqslant_{\gamma_2} \varphi(n)$. Now we have $C \in \gamma_2$ with $\varphi(n) \in C$, $\varphi(m) \notin C$, and pairwise continuity clearly implies $n \in \varphi^{-1}[C] \in \gamma_1$, $m \notin \varphi^{-1}[C]$, which contradicts $m \leqslant_{\gamma_1} n$. Hence $\varphi : (N_1,\mathcal{L}_{N_1}^{\gamma_1}) \to (N_2,\mathcal{L}_{N_2}^{\gamma_2})$ is ω-preserving. On the other hand for $U \in u_2$ we have $\varphi^{\leftarrow}U = \varphi^{-1}[U] \in u_1$, and likewise $\varphi^{\leftarrow}F \in v_1^c$ for $F \in v_2^c$, so $\varphi : (N_1,\mathcal{L}_{N_1}^{\gamma_1},u_1,v_1^c) \to (N_2,\mathcal{L}_{N_2}^{\gamma_2},u_2,v_2^c)$ is bicontinuous. Hence

$$\mathfrak{K}\big((N_1,u_1,v_1) \xrightarrow{\varphi} (N_2,u_2,v_2)\big) = \big(N_1,\mathcal{L}_{N_1}^{\gamma_1},u_1,v_1^c\big) \xrightarrow{\varphi} \big(N_2,\mathcal{L}_{N_2}^{\gamma_2},u_2,v_2^c\big)$$

defines a functor $\mathfrak{K} : \mathbf{Bitop_{w0}} \to \mathbf{ifPDitop_0}$. It is clear from Lemma 3.6(1) that this functor is a variant of the functor with the same name considered in [23] in connection with real dicompactness.

Theorem 3.7. *$\mathfrak{K}$ is a concrete isomorphism between the constructs $\mathbf{Bitop_{w0}}$ and $\mathbf{ifPDitop_0}$.*

Proof. For $(N, \mathcal{L}_N, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{ifPDitop_0}$ the bitopological space (N, τ, κ^c) is weakly pairwise T_0 since $\gamma = \tau \cup \kappa$ is point separating by the T_0 property. It follows easily that

$$\mathfrak{B}\big((N_1, \mathcal{L}_{N_1}, \tau_1, \kappa_1) \xrightarrow{\varphi} (N_2, \mathcal{L}_{N_2}, \tau_2, \kappa_2)\big) = \big((N_1, \tau_1, \kappa_1^c) \xrightarrow{\varphi} (N_2, \tau_2, \kappa_2^c)\big)$$

defines a functor $\mathfrak{B} : \mathbf{ifPDitop_0} \to \mathbf{Bitop_{w0}}$. Moreover, by Lemma 3.6(2) we have $\mathcal{L}_N = \mathcal{L}_N^{\gamma}$ for any plain T_0 texture space $(N, \mathcal{L}_N, \tau, \kappa)$ and $\gamma = \tau \cup \kappa = \tau \cup (\kappa^c)^c$. Hence, $\mathfrak{K} \circ \mathfrak{B} = \mathbf{1}_{\mathbf{ifPDitop_0}}$, and $\mathfrak{B} \circ \mathfrak{K} = \mathbf{1}_{\mathbf{Bitop_{w0}}}$ is immediate from the definitions, so $\mathfrak{K}$ is an isomorphism and $\mathfrak{B}$ its inverse. Since $\mathfrak{K}$ is identity carried, it is a concrete isomorphism between the constructs $\mathbf{Bitop_{w0}}$ and $\mathbf{ifPDitop_0}$. □

Naturally, $\mathbf{ifPDitop_0}$ may be replaced in Theorem 3.7 by either of the isomorphic categories $\mathbf{dfPDitop_0}$ and $\mathbf{fPDitop_0}$.

Examples 3.8. (1) The unit interval texture $(\mathbb{I}, \mathcal{I}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$ is the image under $\mathfrak{K}$ of the bitopological space $(\mathbb{I}, u_{\mathbb{I}}, v_{\mathbb{I}})$, where $u_{\mathbb{I}} = \{[0, r) \mid r \in \mathbb{I}\} \cup \{\mathbb{I}\}$, $v_{\mathbb{I}} = \{(r, 1] \mid r \in \mathbb{I}\} \cup \{\mathbb{I}\}$.

(2) The real texture $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is the image under $\mathfrak{K}$ of the bitopological space $(\mathbb{R}, u_{\mathbb{R}}, v_{\mathbb{R}})$, where $u_{\mathbb{R}} = \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, $v_{\mathbb{R}} = \{(r, \infty) \mid r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

It is now clear why the T_0 axiom on these spaces acts like the weak bitopological T_0 axiom.

We next show that $\mathfrak{K}$ preserves the basic regularity axioms.

Proposition 3.9.

(1) (N, u, v) *is ws* $(ws\,*)$ *if and only if* $(N, \mathcal{L}_N^{\gamma}, u, v^c)$ *is* R_0 *(co-*R_0*).*
(2) (N, u, v) *is pH* $(pH\,*)$ *if and only if* $(N, \mathcal{L}_N^{\gamma}, u, v^c)$ *is* R_1 *(co-*R_1*).*
(3) (N, u, v) *is regular (regular* $*$*) if and only if* $(N, \mathcal{L}_N^{\gamma}, u, v^c)$ *is regular (co-regular).*

Proof. We sketch the proof of the first part of (1). Suppose that (N, u, v) is ws [16] and take $G \in u$, $n \in N$ with $G \not\subseteq Q_n$. Then $n \in G$ so $\overline{n}^v \subseteq G$ by ws. Now $n \in \overline{n}^v \in v^c$, so $[P_n] \subseteq G$ which shows that $(N, \mathcal{L}_N^{\gamma}, u, v^c)$ is R_0. The converse is proved likewise.

The proofs of the remaining results are similar, and are omitted. □

A T_0 space which is R_0 (co-R_0) is called T_1 (co-T_1). We recall the following useful characterizations of these properties from [7, Theorem 4.11]. That for T_1 holds because a plain texture is coseparated by [7, Lemma 4.3].

$(N, \mathcal{L}_N, \tau, \kappa)$ is T_1 if and only if $P_n \in \kappa$ for all $n \in N$.

$(N, \mathcal{L}_N, \tau, \kappa)$ is co-T_1 if and only if $Q_n \in \tau$ for all $n \in N$.

A T_0 space which is R_1 (co-R_1) is called T_2 (co-T_2). The following is a useful characterization of plain bi-T_2 spaces which follows from [7, Theorem 4.17].

$(N, \mathcal{L}_N, \tau, \kappa)$ is bi-T_2 if and only if $m, n \in N$, $n \not\leqslant m \Longrightarrow \exists H \in \tau$, $K \in \kappa$ with $H \subseteq K$, $m \in H$ and $n \notin K$.

Finally, a T_0 space which is regular (coregular) is called T_3 (co-T_3).

We end by defining a particular ditopology on a plain texture which could be of interest.

Definition 3.10. Let $(N, \mathcal{L}_N)$ be a plain texture. The ditopology on $(N, \mathcal{L}_N)$ with subbase $\{Q_n \mid n \in N\}$ and subcobase $\{P_n \mid n \in N\}$ is called the *minimal bi-*T_1 *ditopology on* $(N, \mathcal{L}_N)$.

The above characterizations of T_1 and co-T_1 ditopological texture spaces implies that the minimal bi-T_1 ditopology is indeed the coarsest ditopology on $(N, \mathcal{L}_N)$ which is both T_1 and co-T_1.

Proposition 3.11. *Let* $(N, \leqslant)$ *be a poset and* $(N, \mathcal{L}_N)$ *the corresponding plain texture.*

(1) *If N is finite then the ditopology* (τ, κ) *on* $(N, \mathcal{L}_N)$ *is the minimal bi-*T_1 *ditopology if and only if it is bidiscrete, i.e.* $\tau = \kappa = \mathcal{L}_N$*. In this case the minimal bi-*T_1 *ditopology satisfies all the separation axioms described in* [7].
(2) *If* $(N, \leqslant)$ *is totally ordered and dense in the sense that given* $n_1 < n_2$ *in N there exists* $n \in N$ *with* $n_1 < n < n_2$*, then the minimal bi-*T_1 *ditopology on* $(N, \mathcal{L}_N)$ *is biregular and hence bi-*T_3*.*

Proof. (1) Let (τ, κ) be the minimal bi-T_1 ditopology, and take $L \in \mathcal{L}_N$. Then $L = \bigcap\{Q_n \mid P_n \not\subseteq L\}$ by [5, Theorem 1.2(6)], and so $L \in \tau$ as $N \setminus L$ is finite. Likewise, $L \in \kappa$ by [5, Theorem 1.2(7)] and so $\tau = \kappa = \mathcal{L}_N$. The proof of the converse is similar, and is omitted.

(2) We note first that if $(N, \leqslant)$ is totally ordered then $Q_n = \{n' \in N \mid n \not\leqslant n'\} = \{n' \in N \mid n' < n\}$ (where $n' < n$ means $n' \leqslant n$ and $n' \neq n$), and so $Q_n \subseteq P_n$.

Now let $(N, \leqslant)$ be totally ordered and dense. Then to show the minimal bi-T_1 ditopology is regular [7], take $G \in \tau$ and $n \in N$ with $n \in G$. We must show the existence of $H \in \tau$ with $n \in H$ and $[H] \subseteq G$. By the definition of τ we have $n_1, n_2, \ldots, n_k \in N$ with $n \in Q_{n_1} \cap Q_{n_2} \cap \cdots \cap Q_{n_k} \subseteq G$, but since $(N, \leqslant)$ is totally ordered we may choose the smallest of the elements $n_1, n_2, \ldots, n_k$, say n_i, and then $n \in Q_{n_i} \subseteq G$ by the remark above. Moreover, $n < n_i$ and so we may choose $m \in N$ with $n < m < n_i$. If we set $H = Q_m$ then $H \in \tau$, $n \in H \subseteq P_m \in \kappa$, whence $[H] \subseteq P_m \subseteq Q_{n_i} \subseteq G$, as required. Hence (τ, κ) is regular, and the proof of coregularity is similar and hence omitted. □

It will be recalled that the minimal T_1 topology on an infinite set in the classical sense is never T_2, so Proposition 3.11(2) represents an important difference between the ditopological and topological cases. The important textures $(\mathbb{I}, \mathcal{I})$ and $(\mathbb{R}, \mathcal{R})$ with their usual ditopologies both illustrate this result. On the other hand, the following example shows that the minimal bi-T_1 ditopology is not always bi-T_2.

Example 3.12. Let X be an infinite set with the discrete partial ordering $\leqslant$, that is for $x, x' \in X$, $x \leqslant x' \iff x = x'$. Then clearly $\mathcal{L}_X = \mathcal{P}(X)$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$ for all $x \in X$. If (τ, κ) is the minimal bi-T_1 ditopology it follows that κ consists of X, $\emptyset$ and the finite subsets of X, while τ is the set of complements of such sets. It is now clearly impossible to satisfy the R_1 and co-R_1 axioms, so in particular this space is not T_2 or co-T_2. Clearly the minimal bi-T_1 ditopology corresponds to the minimal T_1 topology in this case, so for the discrete ordering the ditopological and topological properties are the same.

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