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Fixed point theorems on quasi-partial metric spaces

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ABSTRACT

In this paper, the concept of a quasi-partial metric space is introduced, and some general fixed point theorems in quasi-partial metric spaces are proved.

1. Introduction and preliminaries

A partial metric space (PMS) is an attempt to generalize the metric space by replacing the condition $d(x, x) = 0$ with the condition $d(x, x) \leq d(x, y)$ for all x, y in the definition of the metric; it was first introduced by Matthews (see [1,2]). In his initial papers, Matthews discussed not only the general topological properties of partial metric spaces but also some properties of convergence of sequences. Matthews also stated and proved the fixed point theorem of contractive mapping on partial metric spaces: Any mapping T of a complete partial metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$, has a unique fixed point. Very recently, many authors (see, e.g., [3–6,1,2,7–9]) have focused on this subject and have generalized some fixed point theorems from the class of metric spaces to the class of partial metric spaces. In this paper, we introduce quasi-partial metric spaces (QPMSs), and we discuss the existence of fixed points of self-mappings T on quasi-partial metric spaces.

First, we recall the definition of a quasi-metric.

Let $X \neq \emptyset$. A quasi-metric is a function $d: X \times X \to \mathbb{R}^+$ (where $\mathbb{R}^+$ denotes the set of all non-negative real numbers) satisfying

(QM1) $d(x, x) = 0$,
(QM2) $d(x, y) \leq d(x, z) + d(z, y)$,

for all $x, y, z \in X$. The pair (X, d) is called a quasi-metric space.

We next recall the definitions of a PMS and a QPMS.

Let $X \neq \emptyset$. A partial metric (see [1,2]) is a function $p: X \times X \to \mathbb{R}^+$ satisfying

(PM1) $p(x, y) = p(y, x)$ (symmetry),
(PM2) if $0 \leq p(x, x) = p(x, y) = p(y, y)$, then $x = y$ (equality),

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(PM3) $p(x, x) \leq p(x, y)$ (small self-distances)
(PM4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ (triangle inequality),

for all $x, y, z \in X$. The pair (X, p) is called partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x, \varepsilon): x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X: p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A quasi-partial metric is a function $q: X \times X \to \mathbb{R}^+$ satisfying

(QPM1) if $0 \leq q(x, x) = q(x, y) = q(y, y)$, then $x = y$ (equality),
(QPM2) $q(x, x) \leq q(y, x)$ (small self-distances),
(QPM3) $q(x, x) \leq q(x, y)$ (small self-distances),
(QPM4) $q(x, z) + q(y, y) \leq q(x, y) + q(y, z)$ (triangle inequality),

for all $x, y, z \in X$. The pair (X, q) is called a quasi-partial metric space.

Note that, if $q(x, y) = q(y, x)$ for all $x, y \in X$, then (X, q) becomes a partial metric space. It is easy to see that for a partial metric p on X, the function $d_p: X \times X \to \mathbb{R}^+$ defined by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1}$$

is a (usual) metric on X. Analogously for a quasi-partial metric p on X, the function $d_q: X \times X \to \mathbb{R}^+$ defined by

$$d_q(x, y) = q(x, y) + q(y, x) - q(x, x) - q(y, y) \tag{1.2}$$

is a (usual) metric on X.

Lemma 1. *For a quasi-partial metric q on X,*

$$p_q(x, y) = \frac{1}{2}[q(x, y) + q(y, x)] \quad x, y \in X \tag{1.3}$$

is a partial metric on X.

The proof of Lemma 1 is straightforward. Notice that here

$$p_q(x, x) = q(x, x) \tag{1.4}$$

for all $x \in X$.

Proposition 2 (*See [1,2]*)**.** *Let (X, p) be a PMS. Set $q_p(x, y) = p(x, y) - p(x, x)$ for $x, y \in X$. Then (X, q_p) is a QPMS and their topologies coincide (i.e. $\tau_p = \tau_{q_p}$).*

Actually, Proposition 2 tells us more; that is, (X, q_p) is a quasi-metric space due to the fact that $q_p(x, x) = p(x, x) - p(x, x) = 0$.

Example 3 (*See [1,4]*)**.** The pair $(\mathbb{R}^+, p)$ with $p(x, y) = \max\{x, y\}$ is a partial metric space. Note that in this case $d_p(x, y) = |x - y|$.

Example 4 (*See [10]*)**.** Let X be a set, and let $f: X \to [0, 1)$ be an arbitrary one-to-one function. Set $q(x, y) = \max\{f(y) - f(x), 0\}$ for $x, y \in X$. Then q is a quasi-metric.

Some basic concepts on PMSs and QPMSs are defined below.

Definition 5 (*See [3,1,2]*)**.** Let (X, p) be a PMS. Then:

(i) a sequence $\{x_n\} \subset X$ converges to $x \in X$ if and only if

$$p(x, x) = \lim_{n\to\infty} p(x, x_n);$$

(ii) a sequence $\{x_n\} \subset X$ is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite);
(iii) the PMS (X, p) is said to be complete if every Cauchy sequence $\{x_n\} \subset X$ converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$;
(iv) a mapping $f: X \to X$ is said to be continuous at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Definition 6. Let (X, q) be a QPMS. Then:

(i) a sequence $\{x_n\} \subset X$ converges to $x \in X$ if and only if
$$q(x, x) = \lim_{n\to\infty} q(x, x_n) = \lim_{n\to\infty} q(x_n, x);$$
(ii) a sequence $\{x_n\} \subset X$ is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} q(x_n, x_m)$ and $\lim_{n,m\to\infty} q(x_m, x_n)$ exist (and are finite);
(iii) the QPMS (X, q) is said to be complete if every Cauchy sequence $\{x_n\} \subset X$ converges, with respect to τ_q, to a point $x \in X$ such that $q(x, x) = \lim_{n,m\to\infty} q(x_m, x_n) = \lim_{n,m\to\infty} q(x_m, x_n)$;
(iv) a mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Lemma 7 (*See [3,1,2]*)**.**

(A) *A sequence $\{x_n\}$ is Cauchy in a PMS (X, p) if and only if $\{x_n\}$ is Cauchy in the (corresponding) metric space (X, d_p).*
(B) *A PMS (X, p) is complete if and only if the (corresponding) metric space (X, d_p) is complete. Moreover,*
$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{1.5}$$

Regarding the identities (1.1), (1.3) and (1.4), we can conclude the following results.

Lemma 8. *Let (X, q) be a QPMS, let (X, p_q) be the corresponding PMS, and let (X, d_{p_q}) be the corresponding metric space. The following statements are equivalent.*

(A) *The sequence $\{x_n\}$ is Cauchy in (X, q).*
(B) *The sequence $\{x_n\}$ is Cauchy in (X, p_q).*
(C) *The sequence $\{x_n\}$ is Cauchy in (X, d_{p_q}).*

Lemma 9. *Let (X, q) be a QPMS, let (X, p_q) be the corresponding PMS, and let (X, d_{p_q}) be the corresponding metric space. The following statements are equivalent.*

(A) *(X, q) is complete.*
(B) *(X, p_q) is complete.*
(C) *(X, d_{p_q}) is complete.*

Moreover,
$$\begin{aligned}\lim_{n\to\infty} d_{q_p}(x, x_n) = 0 &\Leftrightarrow p_q(x, x) = \lim_{n\to\infty} p_q(x, x_n) = \lim_{n,m\to\infty} p_q(x_n, x_m)\\ &\Leftrightarrow q(x, x) = \lim_{n\to\infty} q(x, x_n) = \lim_{n,m\to\infty} q(x_n, x_m)\\ &= \lim_{n\to\infty} q(x_n, x) = \lim_{n,m\to\infty} q(x_m, x_n).\end{aligned} \tag{1.6}$$

Note that d_{q_p} in Lemmas 8 and 9 is the same as d_q defined by (1.2).
The following lemma is very useful in the proof of the main theorems.

Lemma 10 (*See [6]*)**.** *Let (X, q) be a QPMS. Then the following hold.*

(A) *If $q(x, y) = 0$ then $x = y$.,*
(B) *If $x \neq y$, then $q(x, y) > 0$ and $q(y, x) > 0$.*

Proof. (A). Let $q(x, y) = 0$. By (QPM2) and (QPM3), we have $q(x, x) \le q(x, y) = 0$ and $q(y, y) \le q(x, y) = 0$. Analogously, $q(x, x) \le q(y, x) = 0$ and $q(y, y) \le q(y, x) = 0$. Thus, we have
$$q(x, x) = q(x, y) = q(y, y) = 0 \quad \text{and} \quad q(x, x) = q(y, x) = q(y, y) = 0.$$
Hence, by (QPM1), we have $x = y$.

(B). Suppose that $x \neq y$. By definition, $q(x, y) \ge 0$ and $q(y, x) \ge 0$ for all $x, y \in X$. Assume that $q(x, y) = 0$. By part (A), we have $x = y$, which is a contradiction. Hence, $q(x, y) > 0$ whenever $x \neq y$. Analogously, $q(y, x) > 0$ whenever $x \neq y$. □

If $T : X \to X$ is any map on X, $O(x) = \{x, Tx, T^2x, \ldots\}$ is called the orbit of x. A mapping $G: X \to \mathbb{R}^+$ is T-orbitally lower semi-continuous at x if $\{x_n\}$ is a sequence in $O(x)$ and $\lim x_n = z$ implies $G(z) \le \liminf G(x_n)$.

Many examples of topological spaces are quasi-metrizable, such as the Sorgenfrey line. In [11], Caristi showed the existence of fixed points on certain quasi-metric spaces and other general spaces.

In this paper, some well-known results (see [11,12]) are extended to quasi-partial metric spaces.

2. Fixed point theorems

In this section, we present our main results. We first give the following theorem.

Theorem 11. *Let (X, q) be a quasi-partial metric space, and let $T: X \to X$. Then the following hold.*

(A) *There exists $\phi: X \to \mathbb{R}^+$ such that*

$$q(x, Tx) \le \phi(x) - \phi(Tx) \quad \text{for all } x \in X$$

if and only if

$$\sum_{n=0}^{\infty} q(T^n x, T^{n+1} x) \quad \text{converges for all } x \in X.$$

(B) *There exists $\phi: X \to \mathbb{R}^+$ such that*

$$q(x, Tx) \le \phi(x) - \phi(Tx) \quad \text{for all } x \in O(x)$$

if and only if

$$\sum_{n=0}^{\infty} q(T^n x, T^{n+1} x) \quad \text{converges for all } x \in O(x).$$

Proof. Necessity of (A).

Let $x \in X$, and let

$$q(x, Tx) \le \phi(x) - \phi(Tx).$$

Define the sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$x_0 = x \quad \text{and} \quad x_{n+1} = Tx_n = T^{n+1} x_0, \text{ for all } n = 0, 1, 2, \dots.$$

Set

$$S_n = \sum_{k=0}^{n} q(x_k, x_{k+1}) = \sum_{k=0}^{n} q(T^k x_0, T^{k+1} x_0).$$

Then

$$\begin{aligned} S_n &\le \sum_{k=0}^{n} \left[\phi(T^k x_0) - \phi(T^{k+1} x_0)\right] \\ &= [\phi(x_0) - \phi(Tx_0)] + \cdots + \left[\phi(T^n x_0) - \phi(T^{n+1} x_0)\right] \\ &= \phi(x_0) - \phi(T^{n+1} x_0) \le \phi(x_0) = \phi(x). \end{aligned} \tag{2.1}$$

Thus, (2.1) implies that $\{S_n\}$ is bounded. Also, $\{S_n\}$ is non-decreasing by definition, and hence it is convergent.

Sufficiency of (A).

Define

$$\phi(x) = \sum_{n=0}^{\infty} q(T^n x, T^{n+1} x) \quad \text{and} \quad S_n(x) = \sum_{k=0}^{n} q(T^k x, T^{k+1} x).$$

Then

$$\phi(Tx) = \sum_{n=0}^{\infty} q(T^{n+1} x, T^{n+2} x) \quad \text{and} \quad S_n(Tx) = \sum_{k=0}^{n} q(T^{k+1} x, T^{k+2} x).$$

Using these definitions, we get

$$\begin{aligned} S_n(x) - S_n(Tx) &= \sum_{k=0}^{n} q(T^k x, T^{k+1} x) - \sum_{k=0}^{n} q(T^{k+1} x, T^{k+2} x) \\ &= q(x, Tx) - q(T^{n+1} x, T^{n+2} x). \end{aligned} \tag{2.2}$$

Since $\sum_{n=0}^{\infty} q(T^n x, T^{n+1} x)$ converges for all $x \in X$,

$$\lim_{n\to\infty} S_n(x) = \phi(x) \quad \text{and} \quad \lim_{n\to\infty} q(T^n x, T^{n+1} x) = 0.$$

Letting $n \to \infty$ in (2.2) yields

$$q(x, Tx) = \phi(x) - \phi(Tx),$$

which completes the proof.

Since (B) can be proved analogously, we omit the proof here. □

We give an illustrative example of a quasi-partial metric which is neither a partial metric nor a quasi-metric.

Example 12. Let $X = [0, 1]$. Define

$$q(x, y) = |x - y| + |x|.$$

Clearly, $q(x, y)$ satisfies (QPM1)–(QPM4), and hence is a quasi-partial metric. However, $q(x, y) \neq q(y, x)$ and $q(x, x) \neq 0$, so q is not a partial metric or a quasi-metric. Define $T: X \to X$ as $Tx = \frac{x}{2}$ for all $x \in X$. Then the series $\sum_{n=0}^{\infty} q(T^n x, T^{n+1} x)$ is convergent. Indeed,

$$\begin{aligned}\sum_{n=0}^{\infty} q(T^n x, T^{n+1} x) &= \sum_{n=0}^{\infty} q\left(\frac{x}{2^{n-1}}, \frac{x}{2^n}\right)\\ &= \sum_{n=0}^{\infty} \left|\frac{x}{2^{n-1}} - \frac{x}{2^n}\right| + \left|\frac{x}{2^{n-1}}\right|\\ &= \sum_{n=0}^{\infty} \left|\frac{x}{2^n}\right| + \left|\frac{x}{2^{n-1}}\right|\\ &= \sum_{n=0}^{\infty} \frac{3x}{2^n} = 3x\frac{1}{1-\frac{1}{2}} = 6x.\end{aligned}$$

Then conditions of Theorem 11 are satisfied for $\phi(x) = 6x$.

The next theorem gives conditions for the existence of fixed points of operators on quasi-partial metric spaces.

Theorem 13. *Let (X, q) and (Y, q) be complete quasi-partial metric spaces. Let also $T: X \to X$, $R: X \to Y$, and $\phi: R(X) \to \mathbb{R}^+$. If there exist $x \in X$ and $c > 0$ such that*

$$\max\{q(y, Ty), cq(Ry, RTy)\} \leq \phi(Ry) - \phi(RTy) \tag{2.3}$$

for all $y \in O(x)$, then the following hold.

(A) *$\lim_{n\to\infty} T^n x = z$ exists.*
(B) *$Tz = z$ if and only if $G(x) = q(x, Tx)$ is T-orbitally lower semi-continuous at x.*
(C) *$q(x, T^n x) \leq \phi(Rx)$.*
(D) *If $y \to q(z, y)$ is continuous for $z \in O(x)$, then $q(T^n x, z) \leq \phi(R^n x)$ and $q(x, z) \leq \phi(Rx)$.*

Proof. Proof of (A). Let $x \in X$. Define the sequence $\{x_n\}_{n=1}^{\infty}$ as follows:

$$x_0 = x \quad \text{and} \quad x_{n+1} = Tx_n = T^{n+1}x_0, \text{ for all } n = 0, 1, 2, \ldots.$$

We will show that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Using (QPM4), we obtain

$$\begin{aligned}q(x_n, x_{n+2}) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) - q(x_{n+1}, x_{n+1})\\ &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}),\end{aligned} \tag{2.4}$$

and similarly,

$$\begin{aligned}q(x_n, x_{n+3}) &\leq q(x_n, x_{n+2}) + q(x_{n+2}, x_{n+3}) - q(x_{n+2}, x_{n+2})\\ &\leq q(x_n, x_{n+2}) + q(x_{n+2}, x_{n+3})\\ &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + q(x_{n+2}, x_{n+3}).\end{aligned} \tag{2.5}$$

Upon generalization, we get

$$\begin{aligned}q(x_n, x_m) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m)\\ &\leq q(T^n x, T^{n+1} x) + q(T^{n+1} x, T^{n+2} x) + \cdots + q(T^{m-1} x, T^m x)\\ &= \sum_{k=n}^{m-1} q(T^k x, T^{k+1} x)\end{aligned} \tag{2.6}$$

for $m > n$. Set $S_n(x) = \sum_{k=0}^{n} q(T^k x, T^{k+1}x)$. From (2.3), we have

$$\begin{aligned} q(T^k x, T^{k+1}x) &\le \max\{q(T^k x, T^{k+1}x), cq(RT^k x, RT^{k+1}x)\} \\ &\le \phi(RT^k x) - \phi(RT^{k+1}x) \quad \text{for all } k = 0, 1, \ldots. \end{aligned} \tag{2.7}$$

This implies that

$$S_n(x) \le \sum_{k=0}^{n} [\phi(RT^k x) - \phi(RT^{k+1}x)] = \phi(Rx) - \phi(RT^{n+1}x) \le \phi(Rx). \tag{2.8}$$

Thus, $\sum_{n=0}^{\infty} q(T^n x, T^{n+1}x)$ is convergent. Taking the limit as $n, m \to \infty$ in (2.6), we get

$$\lim_{m,n\to\infty} q(x_n, x_m) = \lim_{m,n\to\infty} (S_{m-1}(x) - S_n(x)) = 0. \tag{2.9}$$

Using similar arguments, one can show that

$$\lim_{m,n\to\infty} q(x_m, x_n) = 0; \tag{2.10}$$

that is, the sequence $\{x_n\}$ is Cauchy in (X, q). Since (X, q) is complete, (X, d_q) is also complete by Lemma 9, and hence $\lim_{n\to\infty} d_q(T^n x, z) = 0$; that is, $\lim_{n\to\infty} T^n x = z$, which completes the proof of (A). Further,

$$\lim_{n\to\infty} q(T^n x, T^{n+1}x) = 0,$$

and hence

$$\lim_{n\to\infty} q(T^n x, T^{n+1}x) = q(z, z) = 0.$$

Proof of necessity of (B).

Assume that $Tz = z$ and that x_n is a sequence in $O(x)$ with $x_n \to z$. Due to (1.6) in Lemma 9,

$$\lim_{n\to\infty} d_q(z, x_n) = 0 \Leftrightarrow q(z, z) = \lim_{n\to\infty} q(z, x_n) = \lim_{n,m\to\infty} q(x_n, x_m). \tag{2.11}$$

Then $G(z) = q(z, Tz) = q(z, z) \le \liminf q(x_n, Tx_n) = \liminf G(x_n)$; that is, G is T-orbitally lower semi-continuous at x.

Proof of sufficiency of (B).

Suppose that $x_n = T^n x \to z$ and that G is T-orbitally lower semi-continuous at x. Then

$$\begin{aligned} 0 \le q(z, Tz) &= G(z) \le \liminf G(x_n) \\ &= \liminf q(T^n x, T^{n+1}x) \\ &= \liminf q(x_{n+1}, x_{n+2}) = q(z, z) = 0. \end{aligned} \tag{2.12}$$

By Lemma 10, we have $Tz = z$.

Proof of (C).

Using (QPM4) and (2.3), we have

$$\begin{aligned} q(x, T^n x) &\le q(x, Tx) + q(Tx, T^2x) + \cdots + q(T^{n-1}x, T^n x) - q(Tx, Tx) - q(T^2x, T^2x) - \cdots - q(T^{n-1}xT^{n-1}x) \\ &\le q(x, Tx) + q(Tx, T^2x) + \cdots + q(T^{n-1}x, T^n x) \\ &\le [\phi(Rx) - \phi(RTx)] + [\phi(RTx) - \phi(RT^2x)] + \cdots + [\phi(RT^{n-1}x) - \phi(RT^n x)] \\ &= \phi(Rx) - \phi(RT^n x) \le \phi(Rx). \end{aligned} \tag{2.13}$$

Proof of (D).

Letting $n \to \infty$ in (2.13) gives $q(x, z) \le \phi(Rx)$. From (2.6), we have $q(x_n, x_m) \le \sum_{k=n}^{m-1} q(T^k x, T^{k+1}x)$ for $m > n$. Note that

$$\begin{aligned} \sum_{k=n}^{m-1} q(T^k x, T^{k+1}x) &\le \sum_{k=n}^{m-1} [\phi(RT^k x) - \phi(RT^{k+1}x)] \\ &= \phi(RT^n x) - \phi(RT^m x) \\ &\le \phi(RT^n x) \end{aligned} \tag{2.14}$$

for $m > n$. Hence,

$$0 \leq q(T^n x, T^m x) \leq \sum_{k=n}^{m-1} q(T^k x, T^{k+1} x) \leq \phi(RT^n x) - \phi(RT^m x) \leq \phi(RT^n x). \tag{2.15}$$

Letting $m \to \infty$ in (2.15) gives $q(T^n x, z) \leq \phi(RT^n x)$, which completes the proof.

Note that the particular case in which $Y = X$ and $R = I$ gives a version of Caristi's Theorem for quasi-partial metric spaces. □

The next two corollaries are special cases of Theorem 13.

Corollary 14. *Let (X, q) be a complete quasi-partial metric space. Let $T: X \to X$ and $\phi: X \to \mathbb{R}^+$. Suppose that there exists $x \in X$ such that*

$$q(y, Ty) \leq \phi(y) - \phi(Ty) \quad \text{for all } y \in O(x).$$

Then the following hold.

(A) $\lim_{n\to\infty} T^n x = z$ *exists.*
(B) $Tz = z$ *if and only if* $G(x) = q(x, Tx)$ *is* T*-orbitally lower semi-continuous at* x*.*
(C) $q(x, T^n x) \leq \phi(x)$*.*
(D) *If* $y \to q(z, y)$ *is continuous for* $z \in O(x)$*, then* $q(T^n x, z) \leq \phi(T^n x)$ *and* $q(x, z) \leq \phi(x)$*.*

Proof. Take $Y = X$ and $R = I$ and $c = 1$ in Theorem 13. □

Corollary 15. *Let (X, q) be a complete quasi-partial metric space, and let $0 < k < 1$. Suppose that $T: X \to X$ and that there exists $x \in X$ such that*

$$q(Ty, T^2 y) \leq kq(y, Ty) \quad \text{for all } y \in O(x). \tag{2.16}$$

Then the following hold.

(A) $\lim_{n\to\infty} T^n x = z$ *exists.*
(B) $Tz = z$ *if and only if* $G(x) = q(x, Tx)$ *is* T*-orbitally lower semi-continuous at* x*.*
(C) $q(x, T^n x) \leq \frac{1}{1-k} q(x, Tx)$*.*

Proof. Set $\phi(y) = \frac{1}{1-k} q(y, Ty)$ for $y \in O(x)$. Let $y = T^n x$ in (2.16). Then $q(T^{n+1}x, T^{n+2}x) \leq kq(T^n x, T^{n+1} x)$ and $q(T^n x, T^{n+1}x) - kq(T^n x, T^{n+1}x) \leq q(T^n x, T^{n+1}x) - q(T^{n+1}x, T^{n+2}x)$.

Thus,

$$q(T^n x, T^{n+1} x) \leq \frac{1}{1-k}[q(T^n x, T^{n+1}x) - q(T^{n+1}x, T^{n+2}x)]$$

or $q(y, Ty) \leq \phi(y) - \phi(Ty)$. (A)–(C) follow immediately from Corollary 14. □

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