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On Fredholm index, transversal approximations and Quillen's geometric complex cobordism of Hilbert manifolds with some applications to flag varieties of loop groups

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Abstract

In [Contemp. Math. 258 (2000) 1–19], by using Fredholm index we developed a version of Quillen's geometric cobordism theory for infinite dimensional Hilbert manifolds. This cobordism theory has a graded group structure under topological union operation and has push-forward maps for complex orientable Fredholm maps. In this work, by using Quinn's Transversality Theorem [Proc. Sympos. Pure. Math. 15 (1970) 213–222], it will be shown that this cobordism theory has a graded ring structure under transversal intersection operation and has pull-back maps for smooth maps. It will be shown that the Thom isomorphism in this theory will be satisfied for finite dimensional vector bundles over separable Hilbert manifolds and the projection formula for Gysin maps will be proved. After we discuss the relation between this theory and classical cobordism, we describe some applications to the complex cobordism of flag varieties of loop groups and we do some calculations.

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1. Preliminaries

1.1. Complex cobordism

Complex bordism theory MU was originally defined by geometric means as bordism classes of maps of stably complex manifolds. For a space X, $MU_q(X)$ is the set of equivalence classes of maps $M \xrightarrow{f} X$ where M is a closed, stably almost complex manifold. It means that M is a compact smooth manifold without boundary of dimension q with TM stably complex.

Two such maps (M, f) and (N, g) are bordant iff their topological union extends to a map $W \to X$ of a compact stably almost complex manifold W of dimension $q+1$, whose boundary is the union of M and N. Here, the stably almost structures on M and N induced by the embedding into W and the original ones are required to be equivalent. The Abelian group structure on $MU_*(X)$ is given by the operation of disjoint union. In [18], Rene Thom gave the homotopy theoretic construction of this group. The details can be found in [17].

The dual cohomology theory MU called complex cobordism was a given a geometric description by D. Quillen. His detailed construction can be found in [14]. For a manifold X of dimension n an element in $MU^{n-q}(X)$ is represented by a smooth proper map $M \to X$ of a (not necessarily compact) manifold M of dimension q together with an equivalence class of complex orientations. Two such maps are cobordant iff they are bordant as maps with complex orientations.

Now we recall the basic properties of multiplicative generalized cohomology theories with complex orientation.

A multiplicative cohomology theory h is a functor from topological spaces to graded rings satisfying the Eilenberg–Steenrod axioms. Details can be found in [4,1]. A multiplicative cohomology theory h is complex oriented if the complex vector bundles are oriented for h. In a complex oriented cohomology theory the Euler class and the Chern classes of complex vector bundles are defined and satisfy the usual properties. Examples of complex oriented cohomology theories are ordinary cohomology, complex K-theory, elliptic cohomology and complex cobordism.

Complex cobordism is the universal complex oriented cohomology theory and so for any such theory h we have canonical map $MU \to h$.

A complex orientation of a proper map of smooth manifolds $f: M \to N$ is a factorization

$$M \xrightarrow{i} \xi \xrightarrow{\pi} N$$

where ξ is a complex vector bundle and i is an embedding with a stably complex normal bundle.

A compact manifold is said to be complex oriented iff the tangent bundle is stably complex.

All details about complex cobordism and multiplicative complex oriented generalized cohomology theories can be found in [1,3,4,14].

1.2. Infinite dimensional manifolds and Pressley–Segal stratifications

The general reference for the section is [13].

LG is the group of all smooth maps from $\mathbb{S}^1$ to compact semi-simple Lie group G. Its essential homogeneous spaces are infinite flag manifold LG/T and based loop space ΩG where T is a maximal torus of G. We are interested to infinite flag manifold LG/T. The cell complexes and the dual stratifications of LG/T are analogically like as G/T. Also we are interested to Grassmannians $\mathrm{Gr}(H)$ of infinite dimensional complex separable Hilbert space H. Its stratifications can be found in [13].

2. The Fredholm index and complex cobordism of Hilbert manifolds

In [14], Quillen gave a geometric interpretation of cobordism groups which suggests a way of defining the cobordism of separable Hilbert manifolds equipped with suitable structure. In order that such a definition be sensible, it ought to reduce to his for finite dimensional manifolds and smooth maps of manifolds and be capable of supporting reasonable calculations for important types of infinite dimensional manifolds such as homogeneous spaces of free loop groups of finite dimensional Lie groups.

2.1. Cobordism of separable Hilbert manifolds

By a manifold, we mean a smooth manifold modelled on a separable Hilbert space; see Lang [9] for details on infinite dimensional manifolds. The facts about Fredholm map can be found in [2].

Definition 2.1. Suppose that $f: X \to Y$ is a proper Fredholm map with even index at each point. Then f is an *admissible complex orientable map* if there is a smooth factorization

$$f: X \xrightarrow{\tilde{f}} \xi \xrightarrow{q} Y,$$

where $q: \xi \to Y$ is a finite dimensional smooth complex vector bundle and $\tilde{f}$ is a smooth embedding endowed with a complex structure on its normal bundle $\nu(\tilde{f})$.

A complex orientation for a Fredholm map f of odd index is defined to be one for the map $(f, \varepsilon): X \to Y \times \mathbb{R}$ given by $(f, \varepsilon)(x) = (f(x), 0)$ for every $x \in X$. At $x \in X$, $\mathrm{index}(f, \varepsilon)_x = (\mathrm{index}\, f_x) - 1$. Also the finite dimensional complex vector bundle ξ in the smooth factorization will be replaced by $\xi \times \mathbb{R}$.

Suppose that f is an admissible complex orientable map. Then since the map f is the Fredholm and ξ is a finite dimensional vector bundle, we see $\tilde{f}$ is also a Fredholm map. By the surjectivity of q,

$$\mathrm{index}\, \tilde{f} = \mathrm{index}\, f - \dim \xi.$$

Before we give a notion of equivalence of such factorizations $\tilde{f}$ of f, we want to give some definitions.

Definition 2.2. Let X, Y be the smooth separable Hilbert manifolds and $F: X \times \mathbb{R} \to Y$ a smooth map. Then we will say that F is an *isotopy* if it satisfies the following conditions:

(1) For every $t \in \mathbb{R}$, the map F_t given by $F_t(x) = F(x,t)$ is an embedding.
(2) There exist numbers $t_0 < t_1$ such that $F_t = F_{t_0}$ for all $t \leqslant t_0$ and $F_t = F_{t_1}$ for all $t \geqslant t_1$.

The closed interval $[t_0, t_1]$ is called a *proper domain* for the isotopy. We say that two embeddings $f : X \to Y$ and $g : X \to Y$ are *isotopic* if there exists an isotopy $F_t : X \times \mathbb{R} \to Y$ with proper domain $[t_0, t_1]$ such that $f = F_{t_0}$ and $g = F_{t_1}$.

Proposition 2.3. (See [9].) *The relation of isotopy between smooth embeddings is an equivalence relation.*

Definition 2.4. Two factorizations $f : X \xrightarrow{\tilde{f}} \xi \xrightarrow{q} Y$ and $f : X \xrightarrow{\tilde{f}'} \xi' \xrightarrow{q'} Y$ are *equivalent* if ξ and ξ' can be embedded as subvector bundles of a vector bundle $\xi'' \to Y$ such that $\tilde{f}$ and $\tilde{f}'$ are isotopic in ξ'' and this isotopy is compatible with the complex structure on the normal bundle. That is, there is an isotopy F such that for all $t \in [t_0, t_1]$, $F_t : X \to \xi''$ is endowed with a complex structure on its normal bundle which matches that of $\tilde{f}$ and $\tilde{f}'$ in ξ'' at t_0 and t_1 respectively.

By Proposition 2.3, we have:

Proposition 2.5. *The relation of equivalence of admissible complex orientability of proper Fredholm maps between separable Hilbert manifolds is an equivalence relation.*

This generalizes Quillen's notion of complex orientability for maps of finite dimensional manifolds. We can also define a notion of cobordism of admissible complex orientable maps between separable Hilbert manifolds. First we recall some ideas on the transversality.

Definition 2.6. Let $f_1 : M_1 \to N$, $f_2 : M_2 \to N$ be smooth maps between Hilbert manifolds. Then f_1 and f_2 are *transverse* at $y \in N$ if

$$df_1(T_{x_1} M_1) + df_2(T_{x_2} M_2) = T_y N$$

whenever $f_1(x_1) = f_2(x_2) = y$. The maps f_1 and f_2 are said to be *transverse* if they are transverse at every point of N.

Lemma 2.7. *Smooth maps $f_i : M_i \to N$ $(i = 1, 2)$ are transverse if and only if $f_1 \times f_2 : M_1 \times M_2 \to N \times N$ is transverse to the diagonal map $\Delta : N \to N \times N$.*

Definition 2.8. Let $f_1 : M_1 \to N$, $f_2 : M_2 \to N$ be transverse smooth maps between smooth Hilbert manifolds. The *topological pullback*

$$M_1 \sqcap_N M_2 = \{(x_1, x_2) \in M_1 \times M_2 \colon\ f_1(x_1) = f_2(x_2)\}$$

is a submanifold of $M_1 \times M_2$ and the diagram

$$\begin{array}{ccc} M_1 \sqcap_N M_2 & \xrightarrow{f_2^*(f_1)} & M_2 \\ \downarrow{\scriptstyle f_1^*(f_2)} & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow{f_1} & N \end{array}$$

is commutative, where the map ${f_i}^*(f_j)$ is pull-back of f_j by f_i.

Definition 2.9. Let $f_i : X_i \to Y$ $(i = 0, 1)$ be admissible complex oriented maps. Then f_0 is *cobordant* to f_1 if there is an admissible complex orientable map $h : W \to Y \times \mathbb{R}$ such that the maps $\varepsilon_i : Y \to Y \times \mathbb{R}$ given by $\varepsilon_i(y) = (y, i)$ for $i = 0, 1$, are transverse to h and the pull-back map ${\varepsilon_i}^* h$ is equivalent to f_i. The cobordism class of $f : X \to Y$ will be denoted by $[X, f]$.

Proposition 2.10. *If $f : X \to Y$ is an admissible complex orientable map and $g : Z \to Y$ a smooth map transverse to f, then the pull-back map*

$$g^*(f) : Z \,\Pi_Y\, X \to Z$$

is an admissible complex orientable map with finite dimensional pull-back vector bundle

$$g^*(\xi) = Z \,\Pi_Y\, \xi = \{(z, v) \in Z \times \xi :\ g(z) = q(v)\}$$

in the factorization of $g^*(f)$, where $q : \xi \to Y$ is the finite-dimensional complex vector bundle in the factorization of f as in Definition 2.1.

The next result was proved in [12] by essentially the same argument as in the finite dimensional situation using the Implicit Function Theorem [9].

Theorem 2.11. *Cobordism is an equivalence relation.*

Definition 2.12. For a separable Hilbert manifold Y, $\mathcal{U}^d(Y)$ is the set of cobordism classes of the admissible complex orientable proper Fredholm maps of index $-d$.

In the above definition, instead of proper maps, closed maps could be used for infinite dimensional Hilbert manifolds, because of the following result of Smale [16].

Theorem 2.13. *When X and Y are infinite dimensional, every closed Fredholm map $X \to Y$ is proper.*

My next result is the following:

Theorem 2.14. *If $f : X \to Y$ is an admissible complex orientable Fredholm map of index d_1 and $g : Y \to Z$ is an admissible complex orientable Fredholm map of index d_2, then $g \circ f : X \to Z$ is an admissible complex orientable Fredholm map with index $d_1 + d_2$.*

Let $g : Y \to Z$ be an admissible complex orientable Fredholm map of index r. By Theorem 2.14, we have *push-forward, or Gysin map*

$$g_* : \mathcal{U}^d(Y) \to \mathcal{U}^{d+r}(Z)$$

given by $g_*([X, f]) = ([X, g \circ f])$.

We show in [12] that it is well-defined. If $g' : Y \to Z$ is a second map cobordant to g then ${g'}_* = g_*$; in particular, if g and g' are homotopic through proper Fredholm maps they induce the same Gysin maps. Clearly, we have $(h \circ g)_* = h_* g_*$ for admissible complex orientable Fredholm maps h, g and $\mathrm{Id}_* = \mathrm{Id}$.

The graded cobordism set $\mathcal{U}^*(Y)$ of the separable Hilbert manifold Y has a group structure given as follows. Let $[X_1, f_1]$ and $[X_2, f_2]$ be cobordism classes. Then $[X_1, f_1] + [X_2, f_2]$ is the class of the map $f_1 \sqcup f_2 : X_1 \sqcup X_2 \to Y$, where $X_1 \sqcup X_2$ is the topological sum (disjoint union) of X_1 and X_2. We show in [12] that this sum is well-defined. As usual, the class of the empty set $\emptyset$ is the zero element of the cobordism set and the negative of $[X, f]$ is itself with the opposite orientation on the normal bundle of the embedding $\tilde{f}$. Then we have:

Theorem 2.15. *The graded cobordism set $\mathcal{U}^*(Y)$ of the admissible complex orientable maps of Y is a graded Abelian group.*

Now we define relative cobordism.

Definition 2.16. If A is a finite dimensional submanifold of Y, the relative cobordism set $\mathcal{U}^*(Y, A)$ is the set of the admissible complex orientable maps of Y whose images lie in $Y - A$.

More generally:

Theorem 2.17. *Let A be a finite dimensional submanifold of Y. Then the relative cobordism set $\mathcal{U}^*(Y, A)$ is a graded Abelian group and there is a homomorphism $\kappa^* : \mathcal{U}^*(Y, A) \to \mathcal{U}^*(Y)$ by $\kappa^*[M \xrightarrow{h} Y] = [M \xrightarrow{h} Y]$ with $h(M) \subset Y - A$.*

If our cobordism functor $\mathcal{U}^*()$ of admissible complex orientable Fredholm maps is restricted to finite dimensional Hilbert manifolds, it agrees Quillen's complex cobordism functor $MU^*()$.

Theorem 2.18. *For finite dimensional separable Hilbert manifolds $A \subset Y$, there is a natural isomorphism*

$$\mathcal{U}^*(Y, A) \cong MU^*(Y, A).$$

3. Transversal approximations, contravariance and cup products

We would like to define a product structure on the graded cobordism group $\mathcal{U}^*(Y)$. Given cobordism classes $[X_1, f_1] \in \mathcal{U}^{d_1}(Y_1)$ and $[X_2, f_2] \in \mathcal{U}^{d_2}(Y_2)$, their external product is

$$[X_1, f_1] \times [X_2, f_2] = [X_1 \times X_2, f_1 \times f_2] \in \mathcal{U}^{d_1+d_2}(Y_1 \times Y_2).$$

Although there is the external product in the category of cobordism of separable Hilbert manifolds, we cannot necessarily define an internal product on $U^*(Y)$ unless Y is a finite dimensional manifold. However, if admissible complex orientable Fredholm map $f_1 \times f_2$

is transverse to the diagonal imbedding $\Delta : Y \to Y \times Y$, then we do have an internal (cup) product

$$[X_1, f_1] \cup [X_2, f_2] = \Delta^*[X_1 \times X_2, f_1 \times f_2].$$

If Y is finite dimensional, then by Thom's Transversality Theorem in [18], every complex orientable map to Y has a transverse approximation, hence the cup product $\cup$ induces a graded ring structure on $\mathcal{U}^*(Y)$. The unit element 1 is represented by the identity map $Y \to Y$ with index 0. However F. Quinn [15] proved the generalization of Thom's Transversality Theorem for separable Hilbert manifolds using smooth transversal approximations of Sard functions in fine topology.

Details about the fine topology, jets and smooth maps space $C^m(W, N)$, $C^\infty(W, N)$ can be found in [10]. In this topology, the derivatives of the difference function between the function g and its approximation g' are bounded. We would like to interpret this approximation in the fine topology. We need some notation to describe this situation.

Definition 3.1. Let X and Y be smooth manifolds. A *k-jet from X to Y* is an equivalence class $[f, x]_k$ of pairs (f, x) where $f : X \to Y$ is a smooth mapping, $x \in X$. The pairs (f, x) and (f', x') are *equivalent* if $x = x'$, f and f' have same Taylor expansion of order k at x in some pair of coordinate charts centered at x and $f(x)$ respectively. We will write $J^k f(x) = [f, x]_k$ and call this the *k-jet of f* at x.

There is an equivalent definition of this equivalence relation: $[f, x]_k = [f', x']_k$ if $x = x'$ and $T^k_x f = T^k_x f'$ where T^k is the kth tangent mapping.

Definition 3.2. For a topological space X,a covering of X is *locally finite* if every point has a neighborhood which intersects only finitely many elements of the covering.

Approximation g' of g in the smooth fine topology means the following. Let $\{L_i\}_{i \in I}$ be a locally finite cover of W. For every open set L_i, there is a bounded continuous map $\varepsilon_i : L_i \to [0, \infty)$ such that for every $x \in L_i$ and $k > 0$,

$$\left\| J^k g(x) - J^k g'(x) \right\| < \varepsilon_i(x).$$

Definition 3.3. Let E be a Banach space. We say that a collection $\mathcal{S}$ of smooth functions $\alpha : E \to \mathbb{R}$ is a *Sard class* if it satisfies the following conditions:

(1) for $r \in \mathbb{R}$, $y \in E$ and $\alpha \in \mathcal{S}$, then the function $x \to \alpha(rx + y)$ is also in the class $\mathcal{S}$,
(2) if $\alpha_n \in \mathcal{S}$, then the rank of differential $D_x(\alpha_1, \ldots, \alpha_n)$ is constant for all x not in some closed finite dimensional submanifold of E.

Definition 3.4. Let $\mathcal{S}$ be a Sard class on E, U open in E, and M a smooth Banach manifold. We define $\mathcal{S}(U, M)$ to be the collection of *Sard functions* $f : U \to M$ such that for each $x \in U$ there is a neighbourhood V of x, functions $\alpha_1, \ldots, \alpha_n \in \mathcal{S}$, and a smooth map $g : W \to M$, where W open in $\mathbb{R}^n$ contains $(\alpha_1, \ldots, \alpha_n)(V)$, all such that $f|V = g \circ (\alpha_1, \ldots, \alpha_n)|V$.

Definition 3.5. The *support* of a function $f : X \to \mathbb{R}$ is the closure of the set of points x such that $f(x) \neq 0$.

From [15], we have:

Theorem 3.6. *E admits a Sard class $\mathcal{S}$ if $\mathcal{S}(E, \mathbb{R})$ contains a function with bounded non-empty support. In particular, the separable Hilbert space admits Sard classes.*

Definition 3.7. A *refinement* of a covering of X is a second covering, each element of which is contained in an element of the first covering.

Definition 3.8. A topological space is *paracompact* if it is Hausdorff, and every open covering has a locally finite open retirement.

Definition 3.9. A smooth *partition of unity* on a manifold X consists of an covering $\{U_i\}$ of X and a system of smooth functions $\psi_i : X \to \mathbb{R}$ satisfying the following conditions:

(1) $\forall x \in X$, we have $\psi_i(x) \geqslant 0$;
(2) the support of ψ_i is contained in U_i;
(3) the covering is locally finite;
(4) for each point $x \in X$, we have

$$\sum_i \psi_i(x) = 1.$$

Definition 3.10. A manifold X will be said to *admit partitions of unity* if it is paracompact, and if, given a locally finite open covering $\{U_i\}$, there exists a partition of unity $\{\psi_i\}$ such that the support of ψ_i is contained in some U_i.

From [9], we have:

Theorem 3.11. *Let X be a paracompact smooth manifold modelled on a separable Hilbert space H. Then X admits smooth partitions of unity.*

From [6]:

Theorem 3.12. *On a separable Hilbert manifold the functions constructed using the partitions of unity form a Sard class.*

The following result was proved by F. Quinn.

Theorem 3.13. *Let H be the smooth separable Hilbert space and let U be an open set in H. If $f : W \to N$ is a smooth proper Fredholm map, then smooth maps transversal to f are dense in $\mathcal{S}(U, N)$ with the C^0 fine topology.*

We will require the Open Embedding Theorem of Eells and Elworthy [5].

Theorem 3.14. *Let M be a smooth manifold modelled on the separable infinite dimensional Hilbert space H. Then M is diffeomorphic to an open subset of H.*

Theorem 3.15. *The space $\mathcal{S}(U,N)$ is dense in $C^0(U,N)$ in the C^0 fine topology.*

Using these techniques, Quinn proves the following result.

Corollary 3.16. *Let M be a smooth separable Hilbert manifold. If $f: W \to N$ is a smooth proper Fredholm map, then smooth maps transversal to f are dense in $C^0(M,N)$ in the C^0 fine topology.*

From [5], we have:

Theorem 3.17. *Let X and Y be two smooth manifold modelled on the separable infinite dimensional Hilbert space H. If there is a homotopy equivalence $\phi: X \to Y$, then ϕ is homotopic to a diffeomorphism.*

Now I will try to tell a detailed explanation of Quinn's very technical work in Theorem 3.13.

Let $f: W \to N$ be a smooth proper Fredholm map and let $g \in \mathcal{S}(U,N)$ specified in Theorem 3.13. Now we are given an arbitrary C^0 fine neighborhood of g in which we want to find a map transversal to f. Since some neighborhood of the image of g is metrizable, we can impose a metric on it and then we can choose a function $\varepsilon: U \to (0,1)$ so that the ε-neighborhood of g in $\mathcal{S}(U,N)$ lies in the given C^0 fine neighborhood. Now we explain the Smale decomposition of the smooth proper Fredholm map $f: W \to N$. If $x \in N$, then there is a coordinate neighborhood Θ: $U \approx H \times \mathbb{R}^k$ about x such that

$$f^{-1}(u) \supseteqq \bigcup_{i=1}^{n} V_i \supseteqq f^{-1}\bigl(\Theta^{-1}\bigl(H \times \mathbb{D}^k\bigr)\bigr)$$

for some sets V_i such that Ψ_i: $V_i \approx H \times W_i, W_i$ open in $\mathbb{R}^m$, and $\Theta \circ f \circ \Psi_i^{-1} = (\pi, f_i): H \times W_i \to H \times \mathbb{R}^k$, where $\mathbb{D}^k$ is the open unit ball in $\mathbb{R}^k$.

Using separability of U, we cover $g(u)$ by $N - f(W)$ and a countable number of sets of the form $\Theta_i^{-1}(H_i \times \mathbb{D}^{k_i})$, where Θ_i: $u_i \approx H_i \times \mathbb{R}^{k_i}$ are coordinate neighborhoods as given in the Smale decomposition of f. We denote the corresponding coordinate neighborhoods in the domain by (V_{i_j}, Ψ_{i_j}), where Ψ_{i_j}: $V_{i_j} \approx H_i \times W_i$, W_i open in $\mathbb{R}^{m_i}$. Let $\{Y_i\}$ be a locally finite refinement of $\{g^{-1}(\Theta_i^{-1}(H_i \times \mathbb{D}^{k_i})), g^{-1}(N - f(W))\}$, and let $\{Z_i\}$ be a subcover such that $Y_i \supseteqq \overline{Z_i}$.

Inductively we will get an approximation of g. Let $g_0 = g$. Given g_i, let g_{i+1} be the approximation defined by Quinn applied the situation $U_1 = U_2 = Y_i, U_3 = Z_i$, W the disjoint union of the W_{i_j} over j, and the C^0 fine topology neighborhood so small that $g_{i+1} = g_i + \frac{\varepsilon}{2^{i+1}} y_i$, and $g_{i+1}(Y_i) \subset \Theta_i^{-1}(H_i \times \mathbb{D}^{k_i})$, where $y_i \in \mathbb{S}^{k_i}$. Since $\{Y_i\}$ is locally finite, $g' = \lim_{i\to\infty} g_i$ is well-defined. g' is an ε-approximation of g and $g' \in \mathcal{S}(U,N)$ and it is transverse to f everywhere. It is interesting that this approximation can be done in the

C^r fine topology. For separable Hilbert manifolds, it can be even done in the smooth C^∞ fine topology. In this case, $g' \in \overline{\mathcal{S}(U, N)}$.

Theorem 3.18. *Let U be an open set in separable infinite dimensional Hilbert space H and let $f : M \to N$ be a proper Fredholm map between separable infinite dimensional Hilbert manifolds M and N. Then the set of maps transverse to f is dense in the closure of Sard function space $\overline{\mathcal{S}(U, N)}$ in the C^∞ fine topology.*

By Corollary 3.16, a smooth map (even continuous map) $g : Z \to Y$ can be deformed to a smooth map $g' : Z \to Y$ by a small correction until it is transverse to an admissible complex orientable map $f : X \to Y$. It is obvious that they are homotopic each other. By definition of Cobordism and Proposition 2.10, the cobordism functor is contravariant for any smooth map between separable Hilbert manifolds.

Theorem 3.19. *Let $f : X \to Y$ be an admissible complex oriented map and let $g : Z \to Y$ be a smooth (may be continuous) map. Then the cobordism class of the pullback $Z \sqcap_Y X \to Z$ depends only on the cobordism class of f, hence there is a map $g^* : \mathcal{U}^d(Y) \to \mathcal{U}^d(Z)$ given by*

$$g^*[X, f] = g'^*[X, f] = \left[Z \sqcap_Y X, g'^*(f)\right],$$

where g' is a smooth ε-approximation of g which is transverse to f. Moreover, g^* depends only on the homotopy class of g.

Now we give the functorial property of $\mathcal{U}^*$ theory.

Theorem 3.20. *Let X, Y and Z be separable Hilbert manifolds. If $Z \xrightarrow{\alpha} Y \xrightarrow{\beta} X$ are smooth functions, then*

$$(\beta \circ \alpha)^* = \alpha^* \beta^* : \mathcal{U}^d(X) \to \mathcal{U}^d(Z).$$

The identity map $\mathrm{Id} : X \to X$ *induces the identity endomorphism* $\mathrm{Id}^* : \mathcal{U}^d(X) \to \mathcal{U}^d(X)$ *for every d.*

Proof. After making them transverse where appropriate we consider the following commutative diagram

$$\begin{array}{ccccc} M'' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & M \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & X \end{array}$$

where both squares are pullback diagrams with transverse (β, f) and (α, f'); the outer square is then also pullback with transverse $(\beta \circ \alpha, f)$. Given an admissible complex oriented map $f : M \to X$, we show that the maps $M' \xrightarrow{f'} Y$ and $M'' \xrightarrow{f''} Z$ are admissible complex orientable and $\alpha^* \beta^*[M \xrightarrow{f} X] = (\beta\alpha)^*[M \xrightarrow{f} X]$. It is clear that $M'' =$

$Z \sqcap_Y M' = Z \sqcap_Y (Y \sqcap_X M)$ as well as $Z \sqcap_X M$ so that $\alpha^*\beta^*[M \xrightarrow{f} X] = (\beta\alpha)^*[M \xrightarrow{f} X]$ by the following commutative diagram

$$\begin{array}{ccccc}
M'' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & M \\
\downarrow{\scriptstyle \tilde{f}''} & & \downarrow{\scriptstyle \tilde{f}'} & & \downarrow{\scriptstyle \tilde{f}} \\
\xi'' & \xrightarrow{\alpha} & \xi' & \xrightarrow{\beta} & \xi \\
\downarrow{\scriptstyle q''} & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\
Z & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & X
\end{array}$$

where the finite dimensional vector bundle $\xi'' = Z \sqcap_Y \xi' = Z \sqcap_Y (Y \sqcap_X \xi)$ as well as $Z \sqcap_X \xi$. □

Let turn back the interior(cup) products in $\mathcal{U}^*$. Given cobordism classes $[X_1, f_1] \in \mathcal{U}^{d_1}(Y_1)$ and $[X_2, f_2] \in \mathcal{U}^{d_2}(Y_2)$, their external product is

$$[X_1, f_1] \times [X_2, f_2] = [X_1 \times X_2, f_1 \times f_2] \in \mathcal{U}^{d_1+d_2}(Y_1 \times Y_2).$$

If admissible complex orientable Fredholm map $f_1 \times f_2$ is transverse to the diagonal imbedding $\Delta : Y \to Y \times Y$, then we do have an internal (cup) product

$$[X_1, f_1] \cup [X_2, f_2] = \Delta^*[X_1 \times X_2, f_1 \times f_2].$$

If the diagonal imbedding $\Delta : Y \to Y \times Y$ is not transverse to smooth proper Fredholm map $f_1 \times f_2 : X_1 \times X_2 \to$, by Quinn's transversality theorem, we can find a smooth ε-approximation Δ' of Δ which is transverse to $f_1 \times f_2$. Then

Theorem 3.21. *If $[X_1, f_1] \in \mathcal{U}^{d_1}(Y_1)$ and $[X_2, f_2] \in \mathcal{U}^{d_2}(Y_2)$, internal (cup) product*

$$[X_1, f_1] \cup [X_2, f_2] = \Delta^*[X_1 \times X_2, f_1 \times f_2] = \Delta'^*[X_1 \times X_2, f_1 \times f_2] \in \mathcal{U}^{d_1+d_2}(Y)$$

where Δ' is a smooth ε-approximation of Δ which is transverse to $f_1 \times f_2$.

The cup product is well-defined and associative.

Then, $\mathcal{U}^*()$ is a multiplicative contravariant functor for smooth functions on the separable Hilbert manifolds. The question of whether it agrees with other cobordism functors such as representable cobordism seems not so easily answered and there is also no obvious dual bordism functor.

In this section, we show how to define Euler classes in complex cobordism for finite dimensional complex vector bundles over separable Hilbert manifolds. In order to do this, we use Sard classes. We know from [7] that global sections of a vector bundle on a smooth separable Hilbert manifold can be constructed using partitions of unity, then all sections are Sard. Given a smooth vector bundle $\pi : E \to B$ over a separable Hilbert manifold B, we know from Theorem 3.14, that B can be embedded as a open subset of a separable Hilbert space H. By Theorem 3.18, we have

Corollary 3.22. *Let $\pi : E \to B$ be a finite dimensional complex vector bundle over a separable Hilbert manifold B and let $i : B \to E$ be the zero-section. Then there is an approximation $\tilde{i}$ of i with $\tilde{i}$ transverse to i.*

Then, we define the Euler class of a finite dimensional complex vector bundle on a separable Hilbert manifold. Note that Theorem 3.19 implies that this Euler class is a well-defined invariant of the bundle π.

Definition 3.23. Let $\pi : \xi \to B$ be a finite dimensional complex vector bundle of dimension d on a separable Hilbert manifold B with zero-section $i : B \to \xi$. The *$\mathcal{U}$-theory Euler class* of ξ is the element

$$\chi(\pi) = i^* i_*(1) \in \mathcal{U}^{2d}(B).$$

We have the following projection formula for the Gysin map.

Theorem 3.24. *Let $f : X \to Y$ be an admissible complex orientable map and let $\pi : \xi \to Y$ be a finite dimensional smooth complex vector bundle of dimension d. Then*

$$\chi(\xi) \cup [X, f] = f_* \chi(f^* \xi).$$

Proof. Let s be a smooth section of π transverse to the zero section $i : Y \to \xi$. Then $Y' = \{y \in Y\colon\ s(y) = i(y)\}$ is a submanifold of complex codimension d and $\chi(\xi) = [Y', j]$, where $j : Y' \to Y$ is the inclusion. Setting

$$X' = f^{-1} Y' = \bigl\{x \in X\colon\ s\bigl(f(x)\bigr) = i\bigl(f(x)\bigr)\bigr\},$$

which is also a submanifold of X of complex codimension d, we have

$$\chi(\xi) \cup [X, f] = [Y', j] \cup [X, f] \tag{1}$$

$$= [X', f_{|X'}]. \tag{2}$$

Now we determine $f_* \chi(f^* E)$. By transversality theorem, f can be deformed to a smooth map f' such that the composite section $s \circ f' : X \to f'^*$ is transverse to the zero section and they agree on X', hence by definition we have $\chi(f^* \xi) = [X', j]$ where $j : X' \to X$ is the inclusion. Hence, $f_* \chi(f^* \xi) = [X', f_{|X'}]$ by definition of the Gysin map f_*. □

Now we need a useful lemma from [15].

Lemma 3.25. *A smooth split submanifold of a smooth separable Hilbert manifold has a smooth tubular neighborhood.*

Let $\pi : \xi \to X$ be a finite dimensional complex vector bundle of dimension d on a separable Hilbert manifold X with zero-section $i : X \to \xi$. The map i is proper so that we have the Gysin map

$$i_* : \mathcal{U}^j(X) \to \mathcal{U}^{j+2d}(\xi, \xi - U)$$

where U is a smooth neighborhood of the zero section.

The map π is not proper. However if U is contained in a tube U^r of finite radius r, then $\pi_{|\overline{U}}$ is proper and we can define

$$\pi_*: \mathcal{U}^{j+2d}(\xi, \xi - U) \to \mathcal{U}^j(X).$$

Since $\pi i = \mathrm{Id}$ we have $\pi_* i_* = \mathrm{Id}$. The composite map $i\pi$ is homotopic to Id_ξ. If $U = U^\circ$ is itself a tube, the homotopy moves on U and we have *Thom isomorphism*

$$\mathcal{U}^{j+2d}(\xi, \xi - U) \cong \mathcal{U}^j(X).$$

4. The relationship between $\mathcal{U}$-theory and MU-theory

In this section we consider the relationship between $\mathcal{U}$-theory and MU-theory. Later we discuss the particular cases of Grassmannians and LG/T.

First we discuss the general relationship between $\mathcal{U}^*(\ )$ and $MU^*(\ )$. Let X be a separable Hilbert manifold. For each proper smooth map $f: M \to X$ where M is a finite dimensional manifold, there is a pullback homomorphism

$$f^*: \mathcal{U}^*(X) \to \mathcal{U}^*(M) = MU^*(M).$$

If we consider all such maps into X, then there is a unique homomorphism

$$\rho: \mathcal{U}^*(X) \to \varprojlim_{M \downarrow X} MU^*(M),$$

where the limit is taken over all proper smooth maps $M \to X$ from finite dimensional manifolds, which form a directed system along commuting diagrams of the form

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array}$$

and hence give rise to an inverse system along induced maps $f^*: MU^*(M_2) \to MU^*(M_1)$ in cobordism.

Let X be a separable Hilbert manifold. Each of the following conjectures appears reasonable and is consistent with examples we will discuss later. We might also hope that surjectivity could be replaced by isomorphism, but we do not have any examples supporting this.

Conjecture 1. *ρ is always a surjection.*

Conjecture 2. *If $\mathcal{U}^{\mathrm{ev}}(X) = 0$ or $\mathcal{U}^{\mathrm{odd}}(X) = 0$, ρ is a surjection.*

Conjecture 3. *If $MU^{\mathrm{ev}}(X) = 0$ or $MU^{\mathrm{odd}}(X) = 0$, ρ is a surjection.*

Now we discuss some important special cases. Let H be a separable complex Hilbert space, with H^n ($n \geqslant 1$) an increasing sequence of finite dimensional subspaces with $\dim H^n = n$ with $H^\infty = \bigcup_{n \geqslant 1} H^n$ dense in H. We use a theorem of Kuiper [8].

Theorem 4.1. *The unitary group $U(H)$ of a separable Hilbert space H is contractible.*

Let $\mathrm{Gr}_n(H)$ be the space of all n-dimensional subspaces of H, which is a separable Hilbert manifold. Then

$$\mathrm{Gr}_n\big(H^\infty\big) = \bigcup_{k \geqslant n} \mathrm{Gr}_n\big(H^k\big)$$

is a dense subspace of $\mathrm{Gr}_n(H)$ which we will take it to be a model for the classifying space $BU(n)$.

Theorem 4.2. *The natural embedding $\mathrm{Gr}_n(H^\infty) \to \mathrm{Gr}_n(H)$ is a homotopy equivalence, and the natural n-plane bundle $\xi_n \to \mathrm{Gr}_n(H)$ is universal.*

Proof. By a theorem of Pressley and Segal [13], the unitary group $U(H)$ acts on $\mathrm{Gr}(H)$ transitively and hence $U(H)$ acts on $\mathrm{Gr}_n(H)$ transitively. Let H^n be an n-dimensional subspace of infinite dimensional separable Hilbert space H and let H' be the orthogonal complement of H^n in H. The stabilizer group of H^n is $U(H^n) \times U(H')$ which acts freely on the contractible space $U(H)$. Hence

$$\begin{aligned}
\mathrm{Gr}_n(H) &= U(H)/\big(U\big(H^n\big) \times U(H')\big) \\
&= B\big(U\big(H^n\big) \times U(H')\big) \\
&= BU\big(H^n\big) \times BU(H').
\end{aligned}$$

By Kuiper's Theorem 4.1, $U(H')$ is contractible, hence so is $BU(H')$. Hence

$$\mathrm{Gr}_n(H) \simeq BU\big(H^n\big) = BU(n).$$

On the other hand,

$$\mathrm{Gr}_n\big(H^\infty\big) = \bigcup_{k \geqslant n} U\big(H^k\big)/\big(U\big(H^n\big) \times U(H'')\big) \subset \mathrm{Gr}_n(H),$$

where H'' is the orthogonal complement of H^n in H^k.

By the construction, the natural n-plane bundle $\xi_n \to \mathrm{Gr}_n(H)$ is universal. Also, the natural bundle $\xi_n^\infty \to \mathrm{Gr}_n(H^\infty)$ is classified by the inclusion $\mathrm{Gr}_n(H^\infty) \to \mathrm{Gr}_n(H)$ and since the latter is universal, this inclusion is a homotopy equivalence. $\square$

In particular, the inclusion of the projective space

$$P\big(H^\infty\big) = \bigcup_{n \geqslant 1} P\big(H^n\big) \subset P(H)$$

is a homotopy equivalence.

Theorem 4.3. *The natural homomorphism*

$$\rho : \mathcal{U}^*\big(P(H)\big) \to \varprojlim_n MU^*\big(P\big(H^n\big)\big) = MU^*\big(P\big(H^\infty\big)\big)$$

is surjective.

Proof. We will show by induction that

$$\mathcal{U}^*\big(P(H)\big) \xrightarrow{i_n^*} MU^*\big(P\big(H^{n+1}\big)\big)$$

is surective for each n. It will suffice to show that $x^i \in \operatorname{im} i_n^*$ for $i = 0, \dots, n$. For $n = 0$, this is trivial.

Now we verify it for $n = 1$. By Theorem 4.2, since the natural line bundle $\lambda \to P(H) \simeq P(H^\infty)$ is universal, the following diagram commutes for each $n \geqslant 1$

$$\begin{array}{ccc} \eta_n = i_n^*(\lambda) & \xrightarrow{i_n^*} & \lambda \\ \big\downarrow{\scriptstyle i_n^*(\lambda)} & & \big\downarrow \\ \mathbb{C}P^n = P(H^{n+1}) & \xrightarrow{i_n} & P(H) \end{array}$$

where $i_n : \mathbb{C}P^n = P(H^{n+1}) \to P(H)$ is the inclusion map. By the compatibility of induced bundles, for $n \geqslant 1$ and the generator $x = \chi(\eta_n) \in MU^*(P(H^{n+1}))$, there exists an Euler class $\tilde{x} = \chi(\lambda) \in \mathcal{U}^2(P(H))$ satisfying $i_n^*(\tilde{x}) = x$, where $i_n : P(H^{n+1}) \to P(H)$ is the inclusion map.

Assume that i_n^* is surjective. Then there are elements

$$y_i \in \mathcal{U}^{2i}\big(P(H)\big), \quad i = 0, \dots, n,$$

such that

$$i_n^* y_i = x^i \in MU^{2i}\big(P\big(H^{n+1}\big)\big).$$

Also,

$$i_{n+1}^* y_i = x^i + z_i x^{n+1} \in MU^{2i}\big(P\big(H^{n+2}\big)\big)$$

where $z_i \in MU_{2(n+1-i)}$.

In particular, let $y_n = [W, f] \in \mathcal{U}^{2n}(P(H))$. Then the following diagram commutes

$$\begin{array}{ccc} f^*(\lambda) & \xrightarrow{f^*} & \lambda \\ \big\downarrow & & \big\downarrow \\ W & \xrightarrow{f} & P(H) \end{array}$$

and there is an Euler class $\chi(f^*(\lambda)) = [W', g] \in \mathcal{U}^2(W)$. Now by Theorem 3.24,

$$y_{n+1} = f_* \chi\big(f^*(\lambda)\big) \in \mathcal{U}^{2n+2}\big(P(H)\big)$$

satisfies

$$i_{n+1}^* y_{n+1} = x^n \chi(\eta_n) = x^{n+1}.$$

Hence, $\operatorname{im} i_{n+1}^*$ contains the MU^*-submodule generated by x^i $(i = 0, \dots, n)$ and so i_{n+1}^* is surjective. This completes the induction.

This shows that the induced homomorphism

$$\rho : \mathcal{U}^*\big(P(H)\big) \to \varprojlim_n MU^*\big(P\big(H^n\big)\big) = MU^*\big(P\big(H^\infty\big)\big)$$

is surjective. $\square$

Note that it is also possible to prove this result by using the projective spaces $P(H^{n\perp}) \subseteq P(H)$ to realize cobordism classes restricting to the classes x^n in $MU^*(P(H^\infty))$.

Next we discuss some geometry of Grassmannians from Pressley and Segal [13], whose ideas and notation we assume. We take for our separable Hilbert space $H = L^2(\mathbb{S}^1; \mathbb{C})$ and let H_+ to be the closure of the subspace of H containing the functions $z^n : z \to z^n$ $(n \geqslant 0)$. Then

$$\mathrm{Gr}_0(H) = \varinjlim_{k \geqslant 1} \mathrm{Gr}(H_{-k,k}),$$

where $\mathrm{Gr}(H_{-k,k})$ is the Grassmannian of the finite dimensional vector space

$$H_{-k,k} = z^{-k} H_+ / z^k H_+.$$

$\mathrm{Gr}_0(H)$ is dense in $\mathrm{Gr}(H)$ and is also known to be homotopic to the classifying space of K-theory, $BU \times \mathbb{Z}$.

Theorem 4.4. *For $n \geqslant 1$, the natural homomorphism*

$$\rho : \mathcal{U}^*\big(\mathrm{Gr}_n(H)\big) \to MU^*\big(\mathrm{Gr}_n(H)\big)$$

is surjective.

Proof. For $k \geqslant n$, the inclusion $i : \mathrm{Gr}_n(H_{-k,k}) \to \mathrm{Gr}_n(H)$ induces a contravariant map

$$\mathcal{U}^*\big(\mathrm{Gr}_n(H)\big) \to \mathcal{U}^*\big(\mathrm{Gr}_n(H_{-k,k})\big) = MU^*\big(\mathrm{Gr}_n(H_{-k,k})\big).$$

For $k \geqslant n$, since $C_S \subset \mathrm{Gr}_n(H_{-k,k})$ is transverse to Σ_S, there exists a stratum $\Sigma_{S'}$ such that

$$\sigma_{S',k} = \big[\mathrm{Gr}_n(H_{-k,k}) \cap \Sigma_{S'} \to \mathrm{Gr}_n(H_{-k,k})\big] \in MU^*\big(\mathrm{Gr}_n(H_{-k,k})\big)$$

are the classical Schubert cells. By an argument using the Atiyah–Hirzebruch spectral sequence and results on Schubert cells in cohomology [11], the cobordism classes $\sigma_{S',k}$ provide generators for the MU^*-module $MU^*(\mathrm{Gr}_n(H_{-k,k}))$. Then i^* is surjective. For each k,

$$MU^{\mathrm{odd}}\big(\mathrm{Gr}_n(H_{-k,k})\big) = 0,$$

hence

$$\begin{aligned}\mathcal{U}^*\big(\mathrm{Gr}_n(H)\big) \to \varprojlim_k MU^*\big(\mathrm{Gr}_n(H_{-k,k})\big) \\ = MU^*\big(\mathrm{Gr}_n\big(H^\infty\big)\big) \\ \cong MU^*\big(\mathrm{Gr}_n(H)\big)\end{aligned}$$

is surjective. □

Theorem 4.5. *For a compact connected semi-simple Lie group G,*

$$\rho : \mathcal{U}^*(LG/T) \to MU^*(LG/T)$$

is surjective.

Proof. As LG/T has no odd dimensional cells,the Atiyah–Hirzebruch spectral sequence for $MU^*(LG/T)$ collapses. Hence it suffices to show that the composition

$$\mathcal{U}^*(LG/T) \to MU^*(LG/T) \to H^*(LG/T, \mathbb{Z})$$

is surjective. Since $H^*(LG/T, \mathbb{Z})$ is generated by the Schubert classes ε^w $(w \in W)$ dual to the Schubert cells C_w, and Σ_w is dual to C_w, the image of the stratum Σ_w under the composition map gives ε^w, establishing the desired surjectivity. □

Similarly, we have

Theorem 4.6. *For a compact connected semi-simple Lie group G,*

$$\rho : \mathcal{U}^*(\Omega G) \to MU^*(\Omega G)$$

is surjective.

5. Cobordism classes related to Pressley–Segal stratifications and some calculations

In this section, we show that the stratifications introduced by Pressley and Segal [13] give rise some further interesting cobordism classes in $\mathcal{U}^*(LG/T)$.

The Grassmannian $\operatorname{Gr} H$ of [13] is a separable Hilbert manifold, and the stratum $\Sigma_S \subseteq \operatorname{Gr} H$ is a locally closed contractible complex submanifold of codimension $\ell(S)$ and the inclusion map $\Sigma_S \to \operatorname{Gr}(H)$ is a proper Fredholm map with index $-\ell(S)$.

Therefore, we have

Theorem 5.1. *The stratum $\Sigma_S \to \operatorname{Gr}(H)$ represents a class in $\mathcal{U}^{2\ell(S)}(\operatorname{Gr}(H))$.*

These strata Σ_S are dual to the Schubert cells C_S in the following sense:

(1) the dimension of C_S is the codimension of Σ_S and
(2) C_S meets Σ_S transversely in a single point, and meets no other stratum the same codimension.

The loop group LG acts via the adjoint representation on the Hilbert space

$$H_{\mathfrak{g}} = L^2(\mathbb{S}^1; \mathfrak{g}_{\mathbb{C}}),$$

where $\mathfrak{g}_{\mathbb{C}}$ is the complexified Lie algebra of G. If $\dim G = n$, we can identify $H_{\mathfrak{g}}$ with H^n and since the adjoint representation is unitary for a suitable Hermitian inner product, this identifies LG with a subgroup of $LU(n)$. Then [13] shows how to identify the based loop group ΩG with a submanifold of $\Omega U(n)$, which can be itself identified with a submanifold of $\operatorname{Gr}(H_{\mathfrak{g}})$.

Then ΩG inherits a stratification with strata Σ_λ indexed by homomorphisms $\lambda : \mathbb{T} \to T$. Each stratum $\Sigma_\lambda \subset \Omega G$ is a locally closed contractible complex submanifold of codimension d_λ, and the inclusion map $\Sigma_\lambda \to \Omega G$ is an admissible Fredholm map. Then

Theorem 5.2. *For each λ, the inclusion $\Sigma_\lambda \to \Omega G$ represents a class in $\mathcal{U}^{2d_\lambda}(\Omega G)$.*

If we restrict to the inverse limit $MU^*(\Omega G)$, these stratum Σ_λ provide the basis elements of $MU^{2d_\lambda}(\Omega G)$. Since $H^*(\Omega G)$ is even graded and MU^* is also even graded, the Atiyah–Hirzebruch spectral sequence

$$H^*(\Omega G; MU^*) \Longrightarrow MU^*(\Omega G)$$

collapses, hence we have an isomorphism

$$\mathcal{U}^*(\Omega G)/\ker\rho \cong MU^*(\Omega G) \cong H^*(\Omega G) \otimes MU^*.$$

For $G = SU(2)$, we have

$$\mathcal{U}^*\big(\Omega SU(2)\big)/\ker\rho \cong MU^*\big(\Omega SU(2)\big) \cong \Gamma_{\mathbb{Z}}(\gamma) \otimes MU^*,$$

where $\Gamma_{\mathbb{Z}}(\gamma)$ is a divided power algebra with the $\mathbb{Z}$-module basis $\gamma^{[n]}$ in each degree $2n$ for $n \geqslant 1$.

Such stratifications also exist for the homogeneous space LG/T.

Theorem 5.3. *For $w \in \widetilde{W}$, the inclusion $\Sigma_w \to LG/T$ represents a class in $\mathcal{U}^{2\ell(w)}(LG/T)$.*

Similarly, if we restrict to the inverse limit $MU^*(LG/T)$, we have an isomorphism

$$\mathcal{U}^*(LG/T)/\ker\rho \cong MU^*(LG/T) \cong H^*(LG/T) \otimes MU^*.$$

For $G = SU(2)$, we have

$$\mathcal{U}^*\big(LSU(2)/T\big)/\ker\rho \cong MU^*\big(LSU(2)/T\big) \cong \Gamma_{\mathbb{Z}}(x_0, x_1)/I_{\mathbb{Z}} \otimes MU^*,$$

where the ideal $I_{\mathbb{Z}}$ is given by

$$I_{\mathbb{Z}} = \left(x_0^{[n]}x_1^{[m]} - \binom{n+m-1}{m}x_0^{[n+m]} - \binom{n+m-1}{n}x_1^{[n+m]}\colon\ m, n \geqslant 1\right)$$

and which has the $\mathbb{Z}$-module basis $\{x_0^{[n]}, x_1^{[n]}\}$ in each degree $2n$ for $n \geqslant 1$.

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