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OSCILLATION THEOREMS FOR CERTAIN EVEN ORDER NONLINEAR DAMPED DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we are concerned with a class of nonlinear damped differential equations of even order. By using the generalized Riccati technique and the integral averaging technique, new oscillation criteria are obtained for every solution of the equations to be oscillatory.

1. Introduction. This paper deals with a class of damped differential equations of even order in the form

$$(1.1)\quad \left(\left|x^{(n-1)}(t)\right|^{\alpha-2}x^{(n-1)}(t)\right)' + p(t)\left|x^{(n-1)}(t)\right|^{\alpha-2}x^{(n-1)}(t) + F\Big(t, x(\tau_{01}(t)),\dots,x(\tau_{0m}(t)),\dots,x^{(n-1)}(\tau_{n-11}(t)),\dots, x^{(n-1)}(\tau_{n-1m}(t))\Big) = 0$$

for $t \geq t_0$, where

(i) n is even and $m \in N$;

(ii) $\alpha > 1$ is a constant;

(iii) $F : [t_0,\infty)\times R^m \times R^n \to R$ is a continuous function;

(iv) $\tau_{ki} : [t_0,\infty) \to R$ is a continuous function and $\lim_{t\to\infty}\tau_{0i}(t) = \infty$, $k = 0,1,\dots,n-1$, $i = 1,2,\dots,m$;

(v) $p : [t_0,\infty) \to [0,\infty)$ is a continuous function and

$$\lim_{t\to\infty}\int_{\bar t}^{t}\left(\exp\left\{-\int_{\bar t}^{s}p(\mu)\,d\mu\right\}\right)^{1/(\alpha-1)}ds = \infty$$

for every $\bar t \geq t_0$.

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By a solution of equation (1.1), we mean a function $x(t) \in C^{n-1}([T_x, \infty), R)$ for some $T_x \geq t_0$ which has the property that $|x^{(n-1)}(t)|^{\alpha-2} \times x^{(n-1)}(t) \in C^1([T_x, \infty), R)$ and satisfies equation (1.1) at all sufficiently large t in $[T_x, \infty)$. Without further mention we will assume throughout that every solution $x(t)$ of (1.1) that is under consideration here is continuable to the right and is nontrivial in the sense that $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$. Such a nontrivial solution of (1.1) is called oscillatory if it has an infinite sequence of zeros clustering at $t = \infty$; otherwise it is said to be nonoscillatory.

The oscillation problem for (1.1) with $\alpha = 2$ and for less general equations has been studied by numerous authors. Indeed, the special case

$$\left(|x'(t)|^{\alpha-2} x'(t)\right)' + F(t, x(g(t))) = 0 \tag{1.2}$$

and other related equations have been the subject of intensive studies in recent years because these equations are natural generalizations of the equation

$$x''(t) + F(t, x(g(t))) = 0. \tag{1.3}$$

For recent contributions, we refer the reader to [**2**–**8**, **13**, **14**, **18**–**21**, **25**, **26**, **31**, **32**, **34**, **37**, **38**] and the references therein.

An important tool in the study of oscillatory behavior of solutions for the differential equations is the averaging technique. This method goes back as far as the classical results of Wintner [**35**], where he gave a sufficient condition for oscillation of the linear equation

$$x''(t) + q(t)x(t) = 0. \tag{1.4}$$

Wintner's condition is

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau)\, d\tau\, ds = \infty.$$

Hartman [**16**] who showed that the above limit cannot be replaced by the super limit and proved that the condition

$$-\infty < \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau)\; d\tau\, ds < \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(\tau)\; d\tau\, ds \le \infty$$

implies that equation (1.4) is oscillatory.

The results of Wintner were improved by Kamenev [**17**] who proved that the condition

$$\limsup_{t\to\infty} \frac{1}{t^{\gamma}} \int_{t_0}^{t} (t-s)^{\gamma}\, q(s)\; ds = \infty$$

for some $\gamma > 1$ is sufficient for the oscillation of equation (1.4). In 1989, Philos [**28**] presented a new oscillation criterion for equation (1.4) involving a Kamenev type condition.

Theorem A. *Let $H : D = \{(t,s) : t \ge s \ge t_0\} \to R$ be a continuous function, such that*

$$H(t,t) = 0 \quad \text{for } t \ge t_0, \quad H(t,s) > 0 \quad \text{for all } (t,s) \in D,$$

and has a continuous and nonnegative partial derivative on D with respect to the second variable. Moreover, let $h : D \to R$ be a continuous function with

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)}, \quad (t,s) \in D.$$

Then equation (1.4) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left\{ H(t,s)\, q(s)\; ds - \frac{1}{4} h^2(t,s) \right\} ds = \infty.$$

The above result of Philos has been generalized and extended for classes of differential equations which are special cases of (1.1), and especially in the absence of damping term [**22, 23, 26, 29, 30, 32**].

Although there is an extensive literature concerning the averaging methods for ordinary differential equations, there is less known about higher order functional differential equations.

In 2001, Agarwal et al. [**3**] considered the nth order differential equation with deviating arguments of the form

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-2}x^{(n-1)}(t)\right)' + F(t,x(g(t))) = 0, \quad n\text{even}. \tag{1.5}$$

They gave some oscillation criteria for this equation which improve and extend several known results established in [**2, 4, 6, 8–13, 24, 27, 37**], one of which is as follows:

Theorem B. *Let*

$$F(t,x)\operatorname{sgn}x \geq q(t)|x|^{\beta-1} \quad \textit{for } x\neq 0 \textit{ and } t\geq t_0$$

hold with $\alpha=\beta$ *where* $\beta>0$ *is a constant and* $q(t)\in C([t_0,\infty),R^+)$ *is a function. If there exist* $\sigma,\rho\in C^1([t_0,\infty),R^+)$, *and a constant* $\theta>1$ *such that*

$$\sigma(t)\leq \inf\{t,g(t)\}, \qquad \lim_{t\to\infty}\sigma(t)=\infty \textit{ and } \sigma'(t)>0$$

for $t\geq t_0$ *and for* $T\geq t_0$,

$$\limsup_{t\to\infty}\int_T^t\left[\rho(s)q(s)-\lambda\theta\frac{(\rho'(s))^\alpha}{(\rho(s)\sigma^{n-2}(s)\sigma'(s))^{\alpha-1}}\right]ds=\infty,$$

where $\lambda=(1/\alpha)^\alpha(2(n-1)!)^{\alpha-1}$, *then equation* (1.5) *is oscillatory.*

The above result is interesting since it makes use of the averaging technique for higher order functional differential equations. Indeed, recently, in this direction some oscillation results have been established by Xu and Xia [**38**] for equation (1.5) improving Theorem B and by Wang [**34**] for equation (1.1) with $\alpha=2$.

Thus, it is a natural problem as to whether similar oscillation results hold for equation (1.1). As far as we know, equation (1.1) has never been the subject of investigations in this direction.

Motivated by the ideas of Agarwal et al. [**3**], Wang [**34**], Wong [**36**] and Tiryaki et al. [**32**], we obtain in this paper several new oscillation criteria for equation (1.1) by using a generalized Riccati substitution and the averaging technique. We extend and improve some earlier criteria by allowing more general means along the lines given in [**36**] for second order differential equations. Our results are also extensions of a number of existing ones for linear equations with damping, and therefore will be of interest in oscillation theory.

2. Preliminaries. In order to discuss our main results, we introduce the general mean and we present some properties, which will be used in the proofs of our results.

Let $D = \{(t,s) : t_0 \le s \le t\}$ denote a subset of R^2, and let $D_1 = \{(t,s) : t_0 \le s < t\}$. Consider a kernel function $K(t,s)$, which is defined, continuous, and sufficiently smooth on D, so that the following conditions are satisfied:

(K_1) $K(t,t) = 0$ and $K(t,s) > 0$ for $(t,s) \in D_1$.

(K_2) $\partial K/\partial s(t,s) \le 0$, $-\partial K/\partial s(t,s) = \lambda(t,s)(K(t,s))^{1/\beta}$ for $(t,s) \in D_1$, where $1/\alpha + 1/\beta = 1$.

(K_3) $\partial^2 K/\partial s \partial t(t,s) = \partial^2 K/\partial t \partial s(t,s)$ for $(t,s) \in D$.

(K_4) $\partial/\partial t(\lambda(t,s)(K(t,s))^{-1/\alpha}) \le 0$ for $(t,s) \in D_1$.

(K_5) For each $s \ge t_0$, $\lim_{t\to\infty} K(t,s) = \infty$, and there exist positive constants k_0, K_0 such that

$$0 < k_0 \le \lim_{t\to\infty} \frac{K(t,s)}{K(t,t_0)} \le K_0 < \infty.$$

A kernel function $K(t,s)$ satisfying $(K_1)-(K_4)$ satisfies the following lemma, which may be proved as in [**36**].

Lemma 2.1. *Let $K(t,s)$ be a continuous kernel function on D satisfying $(K_1) - (K_4)$. If $h \in C[t_0,\infty)$ and $h(s) \ge 0$, then*

$$\frac{1}{K(t,t_0)} \int_{t_0}^{t} K(t,s)\, h(s)\; ds$$

is nondecreasing in t.

Let $\rho \in C^1[t_0,\infty)$ and $\rho(t) > 0$ on $[t_0,\infty)$. We take the integral operator A_τ^ρ, which is defined in [**36**], in terms of $K(t,s)$ and $\rho(s)$ as

$$A_\tau^\rho(h;t) = \int_\tau^t K(t,s)\,h(s)\,\rho(s)\,ds, \quad t \ge \tau \ge t_0, \tag{2.1}$$

where $h \in C[t_0,\infty)$. It is easily seen that A_τ^ρ is linear and positive, and in fact satisfies the following:

$$A_\tau^\rho(\alpha_1 h_1 + \alpha_2 h_2;t) = \alpha_1 A_\tau^\rho(h_1;t) + \alpha_2 A_\tau^\rho(h_2;t), \tag{2.2}$$

$$A_\tau^\rho(h;t) \ge 0 \text{ whenever } h \ge 0, \tag{2.3}$$

$$A_\tau^\rho(h';t) = -K(t,\tau)\,h(\tau)\,\rho(\tau) - A_\tau^\rho\left(\left[-\lambda K^{-1/\alpha} + \frac{\rho'}{\rho}\right]h;t\right). \tag{2.4}$$

Here $h_1,\ h_2,\ h \in C[t_0,\infty)$ and α_1,α_2 are real numbers.

For an arbitrary positive $\xi \in C^1[t_0,\infty)$, define the kernel function

$$K(t,s) = \left(\int_s^t \frac{d\tau}{\xi(\tau)}\right)^m, \quad m > \alpha - 1, \tag{2.5}$$

with $\int_{t_0}^{\infty}(1/\xi(\tau))\,d\tau = \infty$. For example, an important particular case is $\xi(\tau) = \tau^\alpha$, where $\alpha \le 1$ is real. When $\xi(\tau) = 1$ we have $K(t,s) = (t-s)^m$, and when $\xi(\tau) = \tau$ we have $K(t,s) = (\ln(t/s))^m$. It is easily verified that the kernel function (2.5) satisfies $(K_1) - (K_5)$.

The following lemmas will also be needed in the proofs of our results. The first is the well-known Kiguradze's lemma. The second can be found in [**14, 27**].

Lemma 2.2. *Let $u(t) \in C^n([t_0,\infty), R^+)$. If $u^{(n)}(t)$ is of a constant sign and not identically zero on any interval of the form $[t^*,\infty)$, then there exist a $t_4 \ge t_0$ and an integer l, $0 \le l \le n$, with $n+l$ even for $u^{(n)}(t) \ge 0$, or $n+l$ odd for $u^{(n)}(t) \le 0$ such that*

$$l > 0 \quad \textit{implies that} \quad u^{(k)}(t) > 0 \quad \textit{for } t \ge t_4, \quad k = 0,1,2,\dots,l-1$$

and

$$l \le n-1 \quad \textit{implies that} \quad (-1)^{l+k}u^{(k)}(t) > 0 \quad \textit{for } t \ge t_4, \quad k = l, l+1, \dots, n-1.$$

Lemma 2.3. *If the function u is as in Lemma 2.2, and*

$$u^{(n-1)}(t)\, u^{(n)}(t) \le 0 \quad \text{for every } t \ge t_u,$$

then for every θ, $0<\theta<1$, we have

$$u(\theta t) \ge \frac{2^{1-n}}{(n-1)!}\left[\frac{1}{2}-\left|\theta-\frac{1}{2}\right|\right]^{n-1} t^{n-1}\left|u^{(n-1)}(t)\right| \quad \text{for all large } t.$$

Lemma 2.4. *Suppose that the conditions* (i)–(v) *hold. Suppose also that* $\operatorname{sgn} F(t, x_{01}, \dots, x_{0m}, \dots, x_{n-11}, \dots, x_{n-1m}) = \operatorname{sgn} x_{01}$ *for* $x_{0i}x_{01} \ge 0$, $i=1,2,\dots,m$. *Then if $x(t)$ is a nonoscillatory solution of* (1.1), *we have*

$$x(t)\, x^{(n-1)}(t) > 0 \quad \textit{for all large } t.$$

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t)>0$ on $[t_1,\infty)$ for some sufficiently large $t_1 \ge t_0$ since a similar argument holds also for the case when $x(t)$ is eventually negative. As $\lim_{t\to\infty}\tau_{0i}(t)=\infty$, there exists $t_2 \ge t_1$ such that $\tau_{0i}(t) \ge t_1$, $t \ge t_2$, $i=1,2,\dots,m$. Hence, $x(\tau_{0i}(t)) \ge 0$, $t \ge t_2$, $i=1,2,\dots,m$. By (1.1), we obtain

$$\begin{aligned}\left(\left|x^{(n-1)}(t)\right|^{\alpha-2} x^{(n-1)}(t)\right)' + p(t)\left|x^{(n-1)}(t)\right|^{\alpha-2} x^{(n-1)}(t) \\ = -F\big(t, x(\tau_{01}(t)), \dots, x(\tau_{0m}(t)), \dots, \\ x^{(n-1)}(\tau_{n-11}(t)), \dots, x^{(n-1)}(\tau_{n-1m}(t))\big) \le 0.\end{aligned}$$

Multiplying this inequality by $\exp\{\int_{t_0}^t p(\mu)\,d\mu\}$, we obtain

$$\left(\exp\left\{\int_{t_0}^t p(\mu)\, d\mu\right\}\left|x^{(n-1)}(t)\right|^{\alpha-2} x^{(n-1)}(t)\right)' \le 0.$$

Thus, it follows that $\exp\{\int_{t_0}^t p(\mu)\,d\mu\}|x^{(n-1)}(t)|^{\alpha-2}x^{(n-1)}(t)$ is decreasing and $x^{(n-1)}(t)$ is eventually of one sign, that is, either $x^{(n-1)}(t)>0$

for $t \geq t_2$ or there is $t_3 \geq t_2$ such that $x^{(n-1)}(t) < 0$ for $t \geq t_3$. The latter case is impossible, for if $x^{(n-1)}(t) < 0$ for $t \geq t_3$, then on integrating the inequality

$$\exp\left\{\int_{t_0}^{t} p(\mu)\, d\mu\right\}\left(-x^{(n-1)}(t)\right)^{\alpha-1} \geq \exp\left\{\int_{t_0}^{t_3} p(\mu)\, d\mu\right\}(-x^{(n-1)}(t_3))^{\alpha-1}$$

from t_3 to t, we have

$$-x^{(n-2)}(t) + x^{(n-2)}(t_3) \geq (-x^{(n-1)}(t_3)) \int_{t_3}^{t}\left(\exp\left\{-\int_{t_3}^{s} p(\mu)\, d\mu\right\}\right)^{1/(\alpha-1)} ds,$$

so on letting $t \to \infty$ and using the fact that (v), we get a contradiction. □

3. Main results. We are now able to state the main results.

Theorem 3.1. *Let conditions* (i)–(v) *hold, and assume that*

(vi) $F \in C([t_0,\infty) \times R^{m\times n}, R)$ *satisfies the one-sided estimate*

$$F(t, x_{01}, \dots, x_{0m}, \dots, x_{n-11}, \dots, x_{n-1m}) \operatorname{sgn} x_{01} \geq q(t)\left(\sum_{i=1}^{m}|x_{0i}|\right)^{\alpha-1} \text{ for } x_{0i}x_{01} \geq 0, \quad i = 1, 2, \dots, m,$$

where $q \in C([t_0,\infty), R^+)$ *and* $q(t)$ *is not identically zero on any ray* $[t^*,\infty)$;

(vii) *there exist a nonempty subset* $J \subset \{1, 2, \dots, m\}$ *and* $\sigma_j(t) \in C^1([t_0,\infty), R)$, $j \in J$, *such that* $\sigma_j(t) \leq \min\{t, \tau_{0j}(t)\}$, $\lim_{t\to\infty} \sigma_j(t) = \infty$ *and* $\sigma_j'(t) > 0$, $t \geq t_0$, $j \in J$.

Suppose also that $K(t,s)$ *satisfies conditions* (K_1) *and* (K_2), *and that* A_τ^ρ *is defined by* (2.1). *If there exist a positive function* $\rho \in C^1[t_0,\infty)$

and a constant θ, $0 < \theta < 1$, *such that*

$$\limsup_{t\to\infty} \frac{1}{K(t,t_0)} A^{\rho}_{t_0}\left(q - \left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1} \frac{1}{\varphi^{\alpha-1}} \times \left\{\frac{1}{\alpha}\left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right|\right\}^{\alpha} ; t\right) = \infty, \tag{3.1}$$

where

$$M(n,\theta) = \frac{2^{2-n}}{(n-2)!}\left[\frac{1}{2} - \left|\theta - \frac{1}{2}\right|\right]^{n-2}, \qquad \varphi(t) = \sum_{j\in J} \sigma_j^{n-2}(t)\,\sigma_j'(t),$$

then every solution of (1.1) *is oscillatory.*

Proof. Suppose on the contrary that equation (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ on $[t_1,\infty)$ for some sufficiently large $t_1 \geq t_0$ since a similar argument holds also for the case when $x(t)$ is eventually negative. As $\lim_{t\to\infty} \tau_{0i}(t) = \infty$ and $\lim_{t\to\infty} \sigma_j(t) = \infty$, there exists $t_2 \geq t_1$ such that $\tau_{0i}(t) \geq t_1$, $\sigma_j(t) \geq t_1$, $t \geq t_2$, $i = 1,2,\ldots,m$, $j \in J$. Hence, $x(\tau_{0i}(t)) > 0$, $x(\sigma_j(t)) > 0$, $t \geq t_2$, $i = 1,2,\ldots,m$, $j \in J$. By Lemma 2.4, there exists $t_3 \geq t_2$ such that $x^{(n-1)}(t) > 0$ for $t \geq t_3$. From (1.1) we obtain $x^{(n)}(t) \leq 0$ for $t \geq t_3$. Moreover, $q(t) \not\equiv 0$ on any ray $[t^*,\infty)$ ensures that $x^{(n)}(t)$ also has this property. Notice next that the hypotheses of Lemma 2.2 are satisfied on $[t_3,\infty)$, which implies that there exists $t_4 \geq t_3$ such that

$$x'(t) > 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for } t \geq t_4. \tag{3.2}$$

It is easy to check that we can apply Lemma 2.3 for $u = x'$ and conclude that there exists $t_5 \geq t_4$ such that

$$\begin{aligned} x'(\theta\sigma_j(t)) &\geq \frac{2^{2-n}}{(n-2)!}\left[\frac{1}{2} - \left|\theta - \frac{1}{2}\right|\right]^{n-2} \sigma_j^{n-2}(t)\, x^{(n-1)}(\sigma_j(t)) \\ &\geq M(n,\theta)\,\sigma_j^{n-2}(t)\, x^{(n-1)}(t) \end{aligned} \tag{3.3}$$

for $\theta \in (0,1)$, $t \geq t_5$ and $j \in J$. Define

$$w(t) = \left(\frac{x^{(n-1)}(t)}{\sum_{j\in J} x(\theta\sigma_j(t))}\right)^{\alpha-1}. \tag{3.4}$$

Then differentiating (3.4) and making use of (1.1), (3.3) and (vi), it follows that

$$\begin{aligned} w'(t) = &-\frac{F\big(t,x(\tau_{01}(t)),\ldots,x(\tau_{0m}(t)),\ldots,x^{(n-1)}(\tau_{n-11}(t)),\ldots,x^{(n-1)}(\tau_{n-1m}(t))\big)}{\big(\sum_{j\in J}x(\theta\sigma_j(t))\big)^{\alpha-1}} \\ &-\frac{p(t)\big(x^{(n-1)}(t)\big)^{\alpha-1}}{\big(\sum_{j\in J}x(\theta\sigma_j(t))\big)^{\alpha-1}} \\ &-(\alpha-1)\,\theta\frac{\big(x^{(n-1)}(t)\big)^{\alpha-1}\sum_{j\in J}\sigma_j'(t)x'(\theta\sigma_j(t))}{\big(\sum_{j\in J}x(\theta\sigma_j(t))\big)^{\alpha}} \\ \le &-q(t)\,\frac{\left(\sum_{i=1}^{m}x(\tau_{0i}(t))\right)^{\alpha-1}}{\left(\sum_{j\in J}x(\theta\sigma_j(t))\right)^{\alpha-1}} - p(t)\,w(t) \\ &-(\alpha-1)\,\theta M(n,\theta)\,\varphi(t)\,w^{\alpha/(\alpha-1)}(t), \end{aligned}$$

that is,

$$q(t) \le -w'(t) - p(t)\,w(t) - (\alpha-1)\,\theta M(n,\theta)\,\varphi(t)\,w^{\alpha/(\alpha-1)}(t) \tag{3.5}$$

for all $t \ge t_5$. Thus, for all $t \ge t_5$, applying the operator $A_{t_5}^{\rho}$ to (3.5) and using (2.4), we obtain

$$\begin{aligned} A_{t_5}^{\rho}(q;t) \le\, & K(t,t_5)\,\rho(t_5)\,w(t_5) \\ & + A_{t_5}^{\rho}\Bigg(\left[-\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right]w \\ & \qquad - (\alpha-1)\,\theta M(n,\theta)\,\varphi w^{\alpha/(\alpha-1)};t\Bigg) \\ \le\, & K(t,t_5)\,\rho(t_5)\,w(t_5) \\ & + A_{t_5}^{\rho}\Bigg(\left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right|w \\ & \qquad - (\alpha-1)\,\theta M(n,\theta)\,\varphi w^{\alpha/(\alpha-1)};t\Bigg). \end{aligned} \tag{3.6}$$

For given t and s, set

$$F_1(u) := \left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right| u - (\alpha-1)\,\theta M(n,\theta)\,\varphi u^{\alpha/(\alpha-1)}, \quad u > 0.$$

$F_1(u)$ obtains its maximum at

$$u := \left(\frac{1}{\alpha\theta M(n,\theta)}\right)^{\alpha-1} \frac{1}{\varphi^{\alpha-1}} \left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right|^{\alpha-1}$$

and
(3.7)

$$F_1(u) \le F_{1\,\max} = \left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1} \frac{1}{\varphi^{\alpha-1}} \left\{\frac{1}{\alpha}\left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right|\right\}^{\alpha}.$$

Then we get, by using (3.7) in (3.6),

$$\begin{aligned} A_{t_5}^{\rho}(q;t) &\le K(t,t_5)\,\rho(t_5)\,w(t_5) \\ &\quad + A_{t_5}^{\rho}\Bigg(\left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1} \\ &\qquad \times \frac{1}{\varphi^{\alpha-1}} \left\{\frac{1}{\alpha}\left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right|\right\}^{\alpha}; t\Bigg). \end{aligned} \tag{3.8}$$

Hence, for all $t \ge t_5$,

$$\begin{aligned} A_{t_5}^{\rho}\Bigg(q - \left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1} \frac{1}{\varphi^{\alpha-1}} \left\{\frac{1}{\alpha}\left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right|\right\}^{\alpha}; t\Bigg) \\ \le K(t,t_5)\,\rho(t_5)\,w(t_5) \le K(t,t_0)\,\rho(t_5)\,w(t_5). \end{aligned}$$

Take the function

$$\begin{aligned} h(s) = q(s) - \left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1} \frac{1}{\varphi^{\alpha-1}(s)} \\ \times \left\{\frac{1}{\alpha}\left|\lambda(t,s)\,K^{-1/\alpha}(t,s) + \frac{\rho'(s)}{\rho(s)} - p(s)\right|\right\}^{\alpha}; \end{aligned}$$

it is easy to see that, for all $t \ge t_5$,

$$\begin{aligned} A_{t_0}^{\rho}(h;t) &= A_{t_0}^{\rho}(h;t_5) + A_{t_5}^{\rho}(h;t) \\ &\le A_{t_0}^{\rho}(q;t_5) + K(t,t_0)\,\rho(t_5)\,w(t_5). \end{aligned}$$

Therefore,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{K(t,t_0)}A_{t_0}^{\rho}\Bigg(q-\left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1}&\frac{1}{\varphi^{\alpha-1}}\\ \times\left\{\frac{1}{\alpha}\left|\lambda K^{-1/\alpha}+\frac{\rho'}{\rho}-p\right|\right\}^{\alpha}&;t\Bigg)\\ \leq\int_{t_0}^{t_5}q(s)\,\rho(s)\,ds+\rho(t_5)\,w(t_5)&<\infty,\end{aligned}$$

which contradicts condition (3.1). This completes the proof. □

A close look at the proof of Theorem 3.1 reveals that condition (3.1) may be replaced by the conditions

$$\limsup_{t\to\infty}\frac{1}{K(t,t_0)}A_{t_0}^{\rho}(q;t)=\infty \tag{3.9}$$

and

$$\limsup_{t\to\infty}\frac{1}{K(t,t_0)}A_{t_0}^{\rho}\left(\frac{1}{\varphi^{\alpha-1}}\left|\lambda K^{-1/\alpha}+\frac{\rho'}{\rho}-p\right|^{\alpha};t\right)<\infty. \tag{3.10}$$

This leads to the following result.

Corollary 3.1. *Let the conditions of Theorem* 3.1 *be satisfied except that condition* (3.1) *is replaced by* (3.9) *and* (3.10). *Then every solution of* (1.1) *is oscillatory.*

Theorem 3.2. *Let conditions* (i)–(vii) *hold. Suppose also that* $K(t,s)$ *satisfies conditions* (K_1)–(K_5), *and that* A_τ^ρ *is defined by* (2.1). *If there exist a positive function* $\rho\in C^1[t_0,\infty)$ *and nonnegative functions* $\Phi_1,\Phi_2\in C[t_0,\infty)$ *and a positive constant* θ *such that, for* $\tau\geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{K(t,t_0)}A_{\tau}^{\rho}(q;t)\geq\Phi_2(\tau) \tag{3.11}$$

and

$$\lim_{t\to\infty}\frac{1}{K(t,t_0)}A_{\tau}^{\rho}\left(\frac{1}{\varphi^{\alpha-1}}\left|\lambda K^{-1/\alpha}+\frac{\rho'}{\rho}-p\right|^{\alpha};t\right)\leq\Phi_1(\tau), \tag{3.12}$$

where $\varphi(t) = \sum_{j\in J} \sigma_j^{n-2}(t)\sigma_j'(t)$, Φ_1 *and* Φ_2 *satisfy*

(3.13)

$$\lim_{t\to\infty} \frac{1}{K(t,t_0)} A_{t_0}^{\rho}\left(\rho^{-\alpha/(\alpha-1)}\varphi\left[\Phi_2 - \frac{1}{\alpha^\alpha}\left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1}\Phi_1\right]_+^{\alpha/(\alpha-1)};t\right) = \infty,$$

with $[\Phi(t)]_+ = \max\{\Phi(t),0\}$ *and* $0<\theta<1$*, then every solution of* (1.1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 3.1 and return to inequality (3.8). Dividing (3.8) through by $K(t,t_0)$, we obtain

$$\frac{1}{K(t,t_0)} A_{t_5}^{\rho}(q;t) - \frac{1}{K(t,t_0)}\left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1}\frac{1}{\alpha^\alpha} \times A_{t_5}^{\rho}\left(\frac{1}{\varphi^{\alpha-1}}\left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right|^{\alpha};t\right) \le \frac{K(t,t_5)}{K(t,t_0)}\rho(t_5)\,w(t_5). \tag{3.14}$$

Taking lim sup in (3.14) as $t\to\infty$ and noting from (3.11), (3.12) and (K_5) for $t_5 \ge t_0$ that

$$\Phi_2(t_5) - \left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1}\frac{1}{\alpha^\alpha}\Phi_1(t_5) \le K_0\rho(t_5)\,w(t_5),$$

it follows that

$$K_0^{-\alpha/(\alpha-1)}\frac{\varphi(t_5)}{(\rho(t_5))^{\alpha/(\alpha-1)}} \times \left[\Phi_2(t_5) - \left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1}\left(\frac{1}{\alpha}\right)^{\alpha}\Phi_1(t_5)\right]_+^{\alpha/(\alpha-1)} \le \varphi(t_5)\,w^{\alpha/(\alpha-1)}(t_5).$$

To reach a contradiction from the foregoing and condition (3.13), we need to show that

$$\lim_{t\to\infty}\frac{1}{K(t,t_0)}A_{t_5}^{\rho}\left(\varphi w^{\alpha/(\alpha-1)};t\right) < \infty. \tag{3.15}$$

Returning to (3.6) and rearranging, we obtain

$$(3.16)\quad A_{t_5}^{\rho}(q;t)+A_{t_5}^{\rho}\Big((\alpha-1)\,\theta M(n,\theta)\,\varphi\Big\{w^{\alpha/(\alpha-1)} - \frac{1}{(\alpha-1)\,\theta M(n,\theta)}\frac{1}{\varphi}\left|\lambda K^{-1/\alpha}+\frac{\rho'}{\rho}-p\right|w\Big\};t\Big) \leq K(t,t_5)\,\rho(t_5)\,w(t_5)\,.$$

Set

$$a(s)=\frac{1}{(\alpha-1)\,\theta M(n,\theta)}\frac{1}{\varphi(s)}\left|\lambda(t,s)\,(K(t,s))^{-1/\alpha}+\frac{\rho'(s)}{\rho(s)}-p(s)\right|$$

and

$$b(s)=w(s)\,.$$

Using the Young inequality, we get

$$(3.17)\quad \frac{1}{(\alpha-1)\,\theta M(n,\theta)}\frac{1}{\varphi}\left|\lambda K^{-1/\alpha}+\frac{\rho'}{\rho}-p\right|w \leq \frac{1}{\alpha}\left[\left(\frac{1}{(\alpha-1)\,\theta M(n,\theta)}\right)\frac{1}{\varphi}\left|\lambda K^{-1/\alpha}+\frac{\rho'}{\rho}-p\right|\right]^{\alpha} + \frac{(\alpha-1)}{\alpha}w^{\alpha/(\alpha-1)}.$$

Substituting (3.17) into (3.16), we obtain

$$(3.18)\quad \frac{(\alpha-1)\,\theta M(n,\theta)}{\alpha}A_{t_5}^{\rho}\left(\varphi w^{\alpha/(\alpha-1)};t\right) + A_{t_5}^{\rho}\left(q-\frac{1}{\alpha}\left(\frac{1}{(\alpha-1)\,\theta M(n,\theta)}\right)^{\alpha-1}\frac{1}{\varphi^{\alpha-1}}\left|\lambda K^{-1/\alpha}+\frac{\rho'}{\rho}-p\right|^{\alpha};t\right) \leq K(t,t_5)\,\rho(t_5)\,w(t_5)\leq K(t,t_0)\,\rho(t_5)\,w(t_5)\,.$$

As in the proof of Theorem 3.1, it can be easily shown that

$$(3.19)\quad \frac{(\alpha-1)\,\theta M(n,\theta)}{\alpha}A_{t_0}^{\rho}\left(\varphi w^{\alpha/(\alpha-1)};t\right) + A_{t_0}^{\rho}\left(q-\frac{1}{\alpha}\left(\frac{1}{(\alpha-1)\,\theta M(n,\theta)}\right)^{\alpha-1}\frac{1}{\varphi^{\alpha-1}}\left|\lambda K^{-1/\alpha}+\frac{\rho'}{\rho}-p\right|^{\alpha};t\right) \leq K(t,t_0)\left\{\frac{(\alpha-1)\,\theta M(n,\theta)}{\alpha}\int_{t_0}^{t_5}\varphi(s)\,\rho(s)\,w^{\alpha/(\alpha-1)}(s)\;ds + \int_{t_0}^{t_5}q(s)\,\rho(s)\;ds+\rho(t_5)\,w(t_5)\right\}.$$

Dividing (3.19) through by $K(t,t_0)$, we note that by Lemma 2.1, and (3.11), (3.12) and (3.19), the following limits exist and are finite:

$$\lim_{t\to\infty}\frac{1}{K(t,t_0)}A_{t_0}^{\rho}\left(\varphi w^{\alpha/(\alpha-1)};t\right),$$
$$\lim_{t\to\infty}\frac{1}{K(t,t_0)}A_{t_0}^{\rho}\left(\frac{1}{\varphi^{\alpha-1}}\left|\lambda K^{-1/\alpha}+\frac{\rho'}{\rho}-p\right|^{\alpha};t\right).$$

Thus we can take the lim sup in (3.19) as $t\to\infty$ and obtain, by (3.11) and (3.12),

$$\begin{aligned}\limsup_{t\to\infty}&\frac{(\alpha-1)\,\theta M(n,\theta)}{\alpha}\frac{1}{K(t,t_0)}A_{t_0}^{\rho}\left(\varphi w^{\alpha/(\alpha-1)};t\right)\\ &\le\frac{(\alpha-1)\,\theta M(n,\theta)}{\alpha}\int_{t_0}^{t_5}\varphi(s)\,\rho(s)\,(w(s))^{\alpha/(\alpha-1)}\,ds-\Phi_2(t_0)\\ &\quad+\frac{1}{\alpha}\left(\frac{1}{(\alpha-1)\,\theta M(n,\theta)}\right)^{\alpha-1}\Phi_1(t_0)+\rho(t_5)\,w(t_5)\\ &\quad+\int_{t_0}^{t_5}q(s)\,\rho(s)\,ds<\infty.\end{aligned}$$

This gives the desired contradiction to (3.13) and completes the proof. □

Now, we consider the following special case of equation (1.1), namely,

$$\begin{aligned}(3.20)\qquad\left(\left|x^{(n-1)}(t)\right|^{\alpha-2}x^{(n-1)}(t)\right)'+p(t)\left|x^{(n-1)}(t)\right|^{\alpha-2}x^{(n-1)}(t)&\\ +q(t)\,f(x(\tau_{01}(t)))=0,\quad n\quad\text{even},&\end{aligned}$$

where $f\in C^1(R,R)$ is such that $uf(u)>0$, $u\neq0$. We can also prove the following theorems analogous to Theorems 3.1 and 3.2, respectively.

Theorem 3.3. *Let the conditions of Theorem* 3.1 *be satisfied except that the condition* (vi) *is now replaced by*

$$(3.21)\qquad\frac{f'(u)}{|f(u)|^{(\alpha-2)/(\alpha-1)}}\ge\beta_1>0,\quad u\neq0,$$

where β_1 *is a constant. If*

$$\limsup_{t\to\infty} \frac{1}{K(t,t_0)} A_{t_0}^{\rho}\left(q - \left(\frac{(\alpha-1)}{\theta M(n,\theta)\beta_1}\right)^{\alpha-1} \frac{1}{\left(\sigma_1^{n-2}(t)\,\sigma_1'(t)\right)^{\alpha-1}} \times \left\{\frac{1}{\alpha}\left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right|\right\}^{\alpha}; t\right) = \infty,$$

then every solution of (3.20) *is oscillatory.*

Proof. Suppose to the contrary that equation (3.20) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ on $[t_1, \infty)$ for some sufficiently large $t_1 \geq t_0$. It follows, as in the proof of Theorem 3.1, that there exists $t_4 \geq t_1$ such that

$$x'(t) > 0 \quad \text{and} \quad x^{(n-1)}(t) > 0 \quad \text{for } t \geq t_4.$$

Define

$$w(t) = \frac{\left(x^{(n-1)}(t)\right)^{\alpha-1}}{f\left(x\left(\theta\sigma_1(t)\right)\right)}.$$

Then, for $t \geq t_4$, by (3.20) and Lemma 2.3, we find that

(3.22)
$$w'(t) \leq -q(t) - p(t)\,w(t) - \theta M(n,\theta)\,\frac{\sigma_1^{n-2}(t)\,\sigma_1'(t)\left(x^{(n-1)}(t)\right)^{\alpha} f'\left(x\left(\theta\sigma_1(t)\right)\right)}{f^2\left(x\left(\theta\sigma_1(t)\right)\right)}.$$

Because of condition (3.21) and $x'(t) > 0$, (3.22) implies that $w(t)$ satisfies the differential inequality

$$q(t) \leq -w'(t) - p(t)\,w(t) - \beta_1\theta M(n,\theta)\,\sigma_1^{n-2}(t)\,\sigma_1'(t)\,(t)\,w^{\alpha/(\alpha-1)}(t).$$

Then, we can complete the rest of the proof by the procedure of the proof of Theorem 3.1. □

From Theorems 3.2 and 3.3 we present the following theorem. We omit the proof because it is similar to those of the theorems cited above.

Theorem 3.4. *The conclusion of Theorem* 3.2 *remains valid for equation* (3.20), *if condition* (vi) *is replaced by* (3.21).

Finally, we consider the following second order differential equations

$$\left(|x'(t)|^{\alpha-2}x'(t)\right)' + p(t)|x'(t)|^{\alpha-2}x'(t) + F\left(t, x(\tau_{01}(t)), x(\tau_{02}(t)), \ldots, x(\tau_{0m}(t)), x'(\tau_{01}(t)), x'(\tau_{02}(t)), \ldots, x'(\tau_{0m}(t))\right) = 0 \tag{3.23}$$

and

$$\left(r(t)|x'(t)|^{\alpha-2}x'(t)\right)' + p(t)|x'(t)|^{\alpha-2}x'(t) + q(t)f(x(t)) = 0. \tag{3.24}$$

Equation (3.23) is the special case of (1.1) with $n = 2$. We can obtain sharper conditions than the condition (3.1) for equation (3.23), and the oscillation criteria for (3.24) do not depend on the signs of the coefficients p and/or q.

Theorem 3.5. *Suppose that* (ii)–(vii) *hold, and let* $K(t,s)$, ρ *and* A_τ^ρ *be the same as in Theorem* 3.1. *If*

$$\limsup_{t\to\infty} \frac{1}{K(t,t_0)} A_{t_0}^\rho\left(q - \frac{1}{\left(\sum_{j\in J}\sigma_j'\right)^{\alpha-1}}\left\{\frac{1}{\alpha}\left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - p\right|\right\}^\alpha; t\right) = \infty, \tag{3.25}$$

then every solution of (3.23) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t) > 0$ on $[t_1, \infty)$ for some sufficiently large $t_1 \geq t_0$. It follows, as in the proof of Theorem 3.1, that there exists $t_3 \geq t_1$ such that $x'(t) > 0$, $x''(t) \leq 0$, $t \geq t_3$. Define

$$w(t) = \frac{(x'(t))^{\alpha-1}}{\left(\sum_{j\in J} x(\sigma_j(t))\right)^{\alpha-1}}. \tag{3.26}$$

Then differentiating (3.26) and making use of (3.23) and $x'(t) \leq x'(\sigma_j(t))$, $j \in J$, it follows that

$$w'(t) \leq -q(t) - p(t)w(t) - (\alpha-1)\frac{(x'(t))^{\alpha-1}\sum_{j\in J} x'(\sigma_j(t))\sigma_j'(t)}{\left(\sum_{j\in J} x(\sigma_j(t))\right)^{\alpha}}$$
$$\leq -q(t) - p(t)w(t) - (\alpha-1)\frac{(x'(t))^{\alpha}\sum_{j\in J}\sigma_j'(t)}{\left(\sum_{j\in J} x(\sigma_j(t))\right)^{\alpha}},$$

that is,

$$q(t) \leq -w(t) - p(t)w(t) - (\alpha-1)\left(\sum_{j\in J}\sigma_j'(t)\right)w^{\alpha/(\alpha-1)}(t)$$

for all $t \geq t_3$. The rest of the proof is similar to that of Theorem 3.1, so we omit the details. □

Theorem 3.6. *Suppose that* $r, p, q \in C([t_0,\infty), R)$, $r(t) > 0$, $xf(x) > 0$, $f'(x)$ *exists and*

$$\frac{f'(x)}{|f(x)|^{(\alpha-2)/(\alpha-1)}} \geq \beta_2 > 0, \quad x \neq 0, \tag{3.27}$$

and β_2 *is a constant. Let* $K(t,s)$, ρ *and* A_τ^ρ *be the same as in Theorem* 3.1. *If*

$$\limsup_{t\to\infty}\frac{1}{K(t,t_0)}A_{t_0}^{\rho}\left(q - \left(\frac{\alpha-1}{\beta_2}\right)^{\alpha-1} \times r\left\{\frac{1}{\alpha}\left|\lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - \frac{p}{r}\right|\right\}^{\alpha};t\right) = \infty, \tag{3.28}$$

then every solution of (3.24) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t) > 0$ on $[t_1,\infty)$ for some sufficiently large $t_1 \geq t_0$. Define

$$w(t) = \frac{r(t)|x'(t)|^{\alpha-2}x'(t)}{f(x(t))}. \tag{3.29}$$

Then differentiating (3.29) and making use of (3.24) and (3.27), it follows that

$$\begin{aligned} w'(t) &= -q(t) - \frac{p(t)}{r(t)} w(t) \\ &\quad - \frac{f'(x(t))}{|f(x(t))|^{(\alpha-2)/(\alpha-1)}} r^{-1/(\alpha-1)} |w(t)|^{\alpha/(\alpha-1)} \\ &\leq -q(t) - \frac{p(t)}{r(t)} w(t) - \beta_2 r^{-1/(\alpha-1)} |w(t)|^{\alpha/(\alpha-1)} \end{aligned}$$

for all $t \geq t_1$. The rest of the proof is similar to that of Theorem 3.1, so we omit the details. □

Theorem 3.7. *Suppose that* $r, p, q \in C([t_0, \infty), R)$*,* $q(t) \geq 0$*,* $r(t) > 0$*, and* $f \in C(R, R)$ *such that*

$$\frac{f(x)}{x} \geq \beta_3 |x|^{\alpha-2} > 0, \quad x \neq 0, \tag{3.30}$$

and β_3 *is a constant. Let* $K(t, s)$*,* ρ *and* A_τ^ρ *be the same as in Theorem* 3.1*. If*

(3.31)

$$\limsup_{t\to\infty} \frac{1}{K(t, t_0)} A_{t_0}^\rho \left(\beta_3 q - r \left\{ \frac{1}{\alpha} \left| \lambda K^{-1/\alpha} + \frac{\rho'}{\rho} - \frac{p}{r} \right| \right\}^\alpha ; t \right) = \infty,$$

then every solution of (3.24) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution $x(t) > 0$ on $[t_1, \infty)$ for some sufficiently large $t_1 \geq t_0$. Define

$$w(t) = \frac{r(t) |x'(t)|^{\alpha-2} x'(t)}{|x(t)|^{\alpha-2} x(t)}. \tag{3.32}$$

Then differentiating (3.32) and making use of (3.24) and (3.30), it follows that

$$\begin{aligned} w'(t) &= -q(t) \frac{f(x(t))}{|x(t)|^{\alpha-2} x(t)} - \frac{p(t)}{r(t)} w(t) - (\alpha - 1) \frac{r(t) |x'(t)|^\alpha}{|x(t)|^\alpha} \\ &\leq -\beta_3 q(t) - \frac{p(t)}{r(t)} w(t) - (\alpha - 1) r^{-1/(\alpha-1)} |w(t)|^{\alpha/(\alpha-1)} \end{aligned}$$

for all $t \geq t_1$. The rest of the proof is similar to that of Theorem 3.1, so we omit the details. □

Remark 3.1. Our results are presented in a form which is essentially new. We also mention that we do not assume that the functions $\tau_{ki}(t)$, $k = 0, 1, \dots, n-1$, $i = 1, 2, \dots, m$, in (1.1) are of retarded, advanced or mixed type. Hence, the above Theorems 3.1 and 3.5 may hold for ordinary, retarded, advanced or mixed type equations.

Remark 3.2. When $p(t) \equiv 0$ and $F(t, x_{01}, \dots, x_{0m}, \dots, x_{n-11}, \dots, x_{n-1m}) = F(t, x_{01})$, Theorem 3.1 gives Theorem 2.1 in [**38**]. Also, when $\alpha = 2$, Theorems 3.1, 3.5, 3.6 and 3.7 give Theorems 2.1, 2.4, 2.5 and 2.6 in [**34**], respectively.

Remark 3.3. For $n = 2$, $\tau_{01}(t) = t$ and $K(t,s) = (t-s)^m$ from Theorem 3.3, we obtain Theorem 6 in [**1**], with $\psi(x(t)) = 1$ and $r(t) = 1$.

Remark 3.4. In the special case of equation (3.20) with $n = 2$, $p(t) \equiv 0$ and $\tau_{01}(t) = t$, Theorems 3.3 and 3.4 are the same Theorems 3.1 and 3.2 in [**32**] with $\psi(x(t)) = 1$ and $r(t) = 1$, respectively.

Remark 3.5. When $n = 2$ and $\alpha = 2$, Theorem 3.5 extends and improves Theorems 1 and 2 of Rogovchenko [**29**]. Also, Theorem 3.6 gives Theorem 1 in [**36**], respectively.

Remark 3.6. Note that in the special case of equation (3.24) with $f(x) = |x|^{\alpha-2}x$, the condition (3.27) is satisfied. Then Theorem 3.6 gives the main Theorem 2.1 given in [**23**].

Remark 3.7. For $p(t) \equiv 0$, $\rho(t) \equiv 1$ and $K(t,s) = (t-s)$, we may derive Corollary 3.2 in [**33**] from Theorem 3.3.

Remark 3.8. The above Theorems 3.6 and 3.7 extend and improve Theorems A and B of Grace and Lalli [**14**], the theorems of Yeh [**39, 40**], and Theorem 1 of Philos [**28**].

4. Examples. In this section we will show the applications of our oscillation criteria with two examples. We will see that the equations in the examples are oscillatory based on the results in Section 3, though the oscillation cannot be demonstrated by the results in [**1**–**14, 16**–**40**] and most other known criteria.

Example 4.1. Consider the even order nonlinear equation

$$\left(\left|x^{(n-1)}(t)\right|^{\alpha-2} x^{(n-1)}(t)\right)' + \frac{\mu}{t}\left|x^{(n-1)}(t)\right|^{\alpha-2} x^{(n-1)}(t) \tag{4.1}$$
$$+ \frac{\left(1+\cos^2 t\right)}{\left(2+\cos^2 t\right)\left(n/2+\sin^2 t\right)} |x(t-2\pi)|^{\alpha-2}\left(1+\frac{1}{1+(x(t-2\pi))^2}\right)$$
$$x(t-2\pi)\left[1+\sum_{i=1}^{n-1}\left(x^{(i)}(t-2i\pi)\right)^2\right] = 0, \quad t \geq 2n\pi,$$

where n is even, $1<\alpha<\gamma$, $0 \leq \mu \leq \alpha-1$. In fact, let

$$\tau_{01}(t) = (t-2\pi), \qquad \rho(t) = t^{\mu} \quad \text{and} \quad K(t,s) = (t-s)^{\gamma}$$

for $t \geq s \geq t_0$. It is easy to see that the conditions (i)–(vii) hold and

$$\varphi(t) = \sigma_1^{n-2}(t)\,\sigma_1'(t) = (t-2\pi)^{n-2}.$$

Then,

$$\limsup_{t\to\infty} \frac{1}{(t-t_0)^{\gamma}} \int_{t_0}^{t} (t-s)^{\gamma} s^{\mu} \left(\frac{\left(1+\cos^2 s\right)}{\left(2+\cos^2 s\right)\left(n/2+\sin^2 s\right)}\right.$$
$$\left. - \left(\frac{1}{\theta M(n,\theta)}\right)^{\alpha-1} \frac{1}{(s-2\pi)^{(n-2)(\alpha-1)}} \left\{\frac{1}{\alpha}\left|\frac{\gamma}{t-s}+\frac{\mu}{s}-\frac{\mu}{s}\right|\right\}^{\alpha}\right) ds$$
$$\geq \limsup_{t\to\infty} \frac{1}{(t-t_0)^{\gamma}}$$
$$\times \int_{t_0}^{t} \left(\frac{2(t-s)^{\gamma} s^{\mu}}{3n+6} - \frac{\gamma^{\alpha}}{\alpha^{\alpha}(\theta M(n,\theta))^{\alpha-1}} (t-s)^{\gamma-\alpha} s^{\mu}\right) ds = \infty.$$

Consequently, all conditions of Theorem 3.1 are satisfied and hence equation (4.1) is oscillatory. In particular, observe that $x(t) = \cos t$ is an oscillatory solution of (4.1) with $n = 4k+2$, $\alpha = 2$ and $\mu = 0$.

Example 4.2. Consider the differential equation

$$(4.2)\quad \left(\left|x^{(n-1)}(t)\right|^{\alpha-2}x^{(n-1)}(t)\right)' + \frac{\mu}{t}\left|x^{(n-1)}(t)\right|^{\alpha-2}x^{(n-1)}(t) + q(t)\left|x\left(\frac{t}{4}\right)\right|^{\alpha-2}x\left(\frac{t}{4}\right) = 0,$$

where $n \geq 4$ is even, $\alpha > 1$, $0 \leq \mu \leq \alpha_0$, $\alpha_0 := \min\{(n-2)\times(\alpha-1)-\alpha, \alpha-1\}$ and $q \in C([1,\infty), R^+)$, Here, we choose $\tau_{01}(t) = t/4$, $\rho(t) = t^{\mu}$ and $K(t,s) = (t-s)^{\gamma}$ for $t \geq s \geq \tau > 1$, such that

$$\alpha < \gamma, \quad \rho(t)\, q(t) \geq \frac{c}{t^2}, \quad c > 0.$$

It is easy to see that the conditions (i)–(vii) hold and

$$\varphi(t) = \sigma_1^{n-2}(t)\,\sigma_1'(t) = \frac{1}{4}\left(\frac{t}{4}\right)^{n-2}.$$

From Theorem 41 in [**15**] we see that

$$(t-s)^{\gamma} \geq t^{\gamma} - \gamma s t^{\gamma-1}, \quad t \geq s.$$

Then,

$$\limsup_{t\to\infty} \frac{1}{(t-1)^{\gamma}} \int_{\tau}^{t} (t-s)^{\gamma}\rho(s)\, q(s)\, ds \geq \limsup_{t\to\infty} \frac{1}{(t-1)^{\gamma}} \int_{\tau}^{t} \left(t^{\gamma} - \gamma s t^{\gamma}\right) \frac{c}{s^2}\, ds \geq \frac{c}{\tau},$$

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{(t-1)^{\gamma}} \int_{\tau}^{t} (t-s)^{\gamma} s^{\mu} \frac{2^{(2n-2)(\alpha-1)}}{s^{(n-2)(\alpha-1)}} \left|\frac{\gamma}{t-s} + \frac{\mu}{s} - \frac{\mu}{s}\right|^{\alpha} ds \\
&\leq \lim_{t\to\infty} \frac{\gamma^{\alpha} 2^{(2n-2)(\alpha-1)}}{(t-\tau)^{\gamma}} \int_{\tau}^{t} (t-s)^{\gamma-\alpha} s^{\mu-(n-2)(\alpha-1)}\, ds \\
&\leq \lim_{t\to\infty} \frac{\gamma^{\alpha} 2^{(2n-2)(\alpha-1)}}{(t-\tau)^{\gamma}} t^{\gamma-\alpha} \left(1 - \frac{\tau}{t}\right)^{\gamma-\alpha} \int_{\tau}^{t} s^{\mu-(n-2)(\alpha-1)}\, ds = 0.
\end{aligned}$$

Therefore,

$$\phi(s) = \phi_2(s) - \frac{1}{\alpha^\alpha}\frac{1}{(\theta M(n,\theta))^{\alpha-1}}\phi_1(s) = \frac{c}{s}.$$

Hence, we obtain

$$\lim_{t\to\infty}\frac{1}{(t-1)^\gamma}\int_1^t (t-s)^\gamma\, s^{\mu-(\mu\alpha/\alpha-1)+n-2}\frac{1}{2^{2n-2}}\left(\frac{c}{s}\right)^{\alpha/(\alpha-1)} ds$$
$$\geq \lim_{t\to\infty}\frac{c^{\alpha/(\alpha-1)}}{2^{2n-2}(t-1)^\gamma}\int_1^t (t-s)^\gamma\, ds = \infty.$$

Therefore condition (3.13) is satisfied. Consequently, by Theorem 3.2, equation (4.2) is oscillatory.

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