



Approximation Properties of a Certain Nonlinear Durrmeyer Operators

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Abstract. The present paper is concerned with a certain sequence of the nonlinear Durrmeyer operators ND_n , very recently introduced by the author [22] and [23], of the form

$$(ND_n f)(x) = \int_0^1 K_n(x, t, f(t)) dt, \quad 0 \leq x \leq 1, \quad n \in \mathbb{N},$$

acting on Lebesgue measurable functions defined on $[0, 1]$, where

$$K_n(x, t, u) = F_n(x, t) H_n(u)$$

satisfy some suitable assumptions. As a continuation of the very recent papers of the author [22] and [23], we estimate their pointwise convergence to functions f and $\psi \circ |f|$ having derivatives are of bounded (Jordan) variation on the interval $[0, 1]$. Here $\psi \circ |f|$ denotes the composition of the functions ψ and $|f|$. The function $\psi : R_0^+ \rightarrow R_0^+$ is continuous and concave with $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$. This study can be considered as an extension of the related results dealing with the classical Durrmeyer operators.

1. Introduction

Let f be a Lebesgue integrable function defined on $[0, 1]$ and let $\mathbb{N} := \{1, 2, \dots\}$. The classical Durrmeyer operators D_n applied to f are defined as

$$(D_n f)(x) = \int_0^1 f(t) F_n(x, t) dt, \quad 0 \leq x \leq 1 \tag{1}$$

where $F_n(x, t) = (n+1) \sum_{k=0}^n p_{n,k}(x) p_{n,k}(t)$, and $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis. These operators were introduced by Durrmeyer [11] and also let us note that these operators were introduced independently by Lupas [28].

These operators are the integral modification of Bernstein polynomials so as to approximate Lebesgue integrable functions defined on the interval $[0, 1]$. Some remarkable approximation properties of the operators

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(1) are presented by Zeng [32], Derriennic [9] and Gupta [14].

The present paper concerns with the nonlinear counterpart of the well-known Durrmeyer operators introduced and studied by the author [22] and [23].

It is worthwhile to mention the very recent papers of the author [25] and [26], which are dealing with some properties of the nonlinear counterpart of the well-known Bernstein operators in BV and DBV -spaces.

By using the techniques due to Musielak [30], Karsli [22] and [23] introduced the following type nonlinear counterpart of the well-known Durrmeyer operators;

$$(ND_n f)(x) = \int_0^1 K_n(x, t, f(t)) dt, \quad 0 \leq x \leq 1, \quad n \in \mathbb{N}, \quad (2)$$

acting on Lebesgue measurable functions on the interval $[0, 1]$, where $K_n(x, t, u)$ satisfy some suitable assumptions. In particular, he proved some existence and approximation theorems for the nonlinear Durrmeyer operators. In particular, in [22] he investigated the pointwise convergence of (2) to the functions of bounded (Jordan) variation on the interval $[0, 1]$ and in [23] he obtained some pointwise convergence for the nonlinear sequence of Durrmeyer operators (2) to the Lebesgue point x of f , as $n \rightarrow \infty$.

Approximation with nonlinear integral operators of convolution type was introduced by J. Musielak in [30] and widely developed in [1]. Especially, nonlinear integral operators of type

$$(T_\lambda f)(x) = \int_a^b K_\lambda(t-x, f(t)) dt, \quad x \in (a, b),$$

and its special cases were studied by Bardaro, Karsli and Vinti [2]-[4], Karsli [18], [20] and Karsli-Gupta [19] in some Lebesgue spaces, BV -spaces and BV_φ -spaces.

As a continuation of the very recent papers of the author [22] and [23], we estimate their pointwise convergence to functions f and $\psi \circ |f|$ having derivatives are of bounded (Jordan) variation on the interval $[0, 1]$.

For future studies, it is important to say that this kind of approach can apply for several linear positive operators presented in the papers [10], [13], [29] and in the very recent books of Gupta-Agarwal [15] (see also Gal [12]).

An outline of the paper is as follows:

The next section contains basic definitions and notations.

In Section 3, the main approximation results of this study are given. They are dealing with the rate of pointwise convergence of the nonlinear Durrmeyer operators ND_n for functions with derivatives of bounded variation on $[0, 1]$. At the point x , which is a discontinuity of the first kind of f and of its derivative, we shall prove that $(ND_n f)(x)$ converge to the limit $f(x)$. Some important papers on this topic for the linear counterpart of positive linear operators are [5]-[16], [19], [24] and [31].

In Section 4, we give some certain results which are necessary to prove the main result.

The final section, that is Section 5, concerns with the proof of the main results presented in Section 3.

2. Preliminaries

In this section, we assemble the main definitions and notations given in [22] and [23].

Let X be the set of all bounded Lebesgue measurable functions $f : [0, 1] \rightarrow \mathbb{R}$.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ such that the function ψ is a continuous and non-decreasing with $\psi(0) = 0, \psi(u) > 0$ for $u > 0$.

We now introduce a sequence of functions. Let $\{K_n(x, t, u)\}_{n \in \mathbb{N}}$ be a sequence of functions $K_n(x, t, u)$ defined by

$$K_n(x, t, u) = (n + 1) \sum_{k=0}^n p_{n,k}(x) p_{n,k}(t) H_n(u), \tag{3}$$

where $H_n : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $H_n(0) = 0$ and $p_{n,k}(x)$ is the Bernstein basis.

Throughout the paper we assume that $\mu : \mathbb{N} \rightarrow \mathbb{R}^+$ is an increasing and continuous function such that $\lim_{n \rightarrow \infty} \mu(n) = \infty$.

First of all we assume that the following conditions hold:

a) $H_n : \mathbb{R} \rightarrow \mathbb{R}$ is such that

$$|H_n(u) - H_n(v)| \leq \psi(|u - v|), \quad \psi \in \Psi,$$

holds for every $u, v \in \mathbb{R}$ and for every $n \in \mathbb{N}$. That is, H_n satisfies a $(L - \psi)$ Lipschitz condition.

b) We now set

$$F_n(x, t) := (n + 1) \sum_{k=0}^n p_{n,k}(x) p_{n,k}(t).$$

c) Denoting by $r_n(u) := H_n(u) - u, u \in \mathbb{R}$ and $n \in \mathbb{N}$, such that

$$\lim_{n \rightarrow \infty} |r_n(u)| = 0$$

uniformly with respect to u .

In other words, for n sufficiently large

$$\sup_u |r_n(u)| = \sup_u |H_n(u) - u| \leq \frac{1}{\mu(n)},$$

holds.

The symbol $[a]$ will denote the greatest integer not greater than a .

3. Convergence Results

Let us take the following type nonlinear Durrmeyer operators,

$$(ND_n f)(x) = \int_0^1 K_n(x, t, f(t)) dt,$$

where

$$K_n(x, t, f(t)) = (n + 1) \sum_{k=0}^n p_{n,k}(x) p_{n,k}(t) H_n(f(t)) = F_n(x, t) H_n(f(t)).$$

We will consider that this operator defined for every $f \in \text{Dom } ND_n$, where $\text{Dom } ND_n$ is the subset of X on which ND_n is well-defined.

As before, we let

$$f_x(t) = \begin{cases} f(t) - f(x+) & , \quad x < t \leq 1 \\ 0 & , \quad t = x \\ f(t) - f(x-) & , \quad 0 \leq t < x \end{cases} \quad (4)$$

and $\bigvee_0^1 \psi(|f_x|)$ is the total variation of $\psi(|f_x|)$ on $[0, 1]$.

We are now ready to establish the main results of this study:

Let $DBV(I)$ denotes the class of differentiable functions defined on a set $I \subset \mathbb{R}$, whose derivatives are bounded variation on I and will be denoted as

$$DBV(I) = \{f : f' \in BV(I)\}.$$

Theorem 1. Let $\psi \in \Psi$ and f be a function with derivatives of bounded variation on $[0, 1]$. Then for every $x \in (0, 1)$, we have for sufficiently large n ,

$$\begin{aligned} |(ND_n f)(x) - f(x)| &\leq \left| \frac{f'(x+) - f'(x-)}{2} \right| \sqrt{\frac{2nx(1-x) + 2}{n^2}} \\ &\quad + \frac{2(n+1)}{n^2x(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (f'_x) + \frac{1}{\mu(n)}, \end{aligned} \quad (5)$$

where $\bigvee_a^b (f'_x)$ is the total variation of f'_x on $[a, b]$.

Theorem 2. Let $\psi \in \Psi$ and $f \in X$ be such that $\psi o |f| \in DBV([0, 1])$. Then for every $x \in (0, 1)$, we have for sufficiently large n ,

$$\begin{aligned} |(ND_n f)(x) - f(x)| &\leq \frac{(\psi o |f|)'(x-) - (\psi o |f|)'(x+)}{2} \sqrt{\frac{2nx(1-x) + 2}{n^2}} \\ &\quad + \frac{2(n+1)}{n^2x(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (\psi o |f|)'_x + \frac{1}{\mu(n)}. \end{aligned}$$

4. Auxiliary Result

In this section we recall some certain results, which are necessary to prove our theorems.

Lemma 1 ([27]). For $(D_n t^s)(x)$, $s = 0, 1, 2$, one has

$$\begin{aligned} (D_n 1)(x) &= 1, \quad (D_n t)(x) = x + \frac{1-2x}{n+2} \\ (D_n t^2)(x) &= x^2 + \frac{[4n - 6(n+1)x]}{(n+1)(n+2)}x + \frac{2}{(n+2)(n+3)}. \end{aligned}$$

For proof of this Lemma see Lorentz [27].

By direct calculation, we find;

$$(D_n (t-x)^2)(x) \leq \frac{2nx(1-x) + 2}{n^2}, \quad (D_n (t-x))(x) = \frac{1-2x}{n+2}.$$

Lemma 2 ([22] and [23]). For all $x \in (0, 1)$ and for each $n \in \mathbb{N}$, one has

$$\lambda_n(x, z) =: \int_0^z F_n(x, u) du \leq \frac{B(x)}{(x - z)^\beta n^{\gamma/\beta}}, \quad 0 \leq z < x, \tag{6}$$

and

$$1 - \lambda_n(x, z) = \int_z^1 F_n(x, u) du \leq \frac{B(x)}{(z - x)^\beta n^{\gamma/\beta}}, \quad x < z < 1. \tag{7}$$

5. Proof of the Theorems

Proof of Theorem 1. In general, a singular integral operator may be written in the form

$$(T_n f)(x) = \int_a^b f(t) K_n(x, t) dt, \tag{8}$$

where $K_n(x, t)$ is the kernel function, defined for $a \leq x, t \leq b$, which the property that for functions $f(x)$ of a certain class in a certain sense, $(T_n f)(x)$ converges to $f(x)$ as $n \rightarrow \infty$.

We can write the difference between $(ND_n f)(x)$ and $f(x)$ as a singular Stieltjes integral as follows;

$$\begin{aligned} (ND_n f)(x) - f(x) &= \int_0^1 H_n(f(t)) F_n(x, t) dt - f(x) \\ &= \int_0^1 [H_n(f(t)) - f(t)] F_n(x, t) dt + \int_0^1 [f(t) - f(x)] F_n(x, t) dt = I_{n,1}(x) + I_{n,2}(x). \end{aligned}$$

Firstly, we consider

$$I_{n,2}(x) = \int_0^1 [f(t) - f(x)] F_n(x, t) dt. \tag{9}$$

Since $f(t) \in DBV[0, 1]$ we can rewrite (9) as follows:

$$\begin{aligned} I_{n,2}(x) &= \int_0^x [f(t) - f(x)] F_n(x, t) dt + \int_x^1 [f(t) - f(x)] F_n(x, t) dt \\ &= - \int_0^x \left[\int_t^x f'(u) du \right] F_n(x, t) dt + \int_x^1 \left[\int_x^t f'(u) du \right] F_n(x, t) dt \\ &= -I_1(x) + I_2(x), \end{aligned}$$

where

$$I_1(x) = \int_0^x \left[\int_t^x f'(u) du \right] F_n(x, t) dt \tag{10}$$

and

$$I_2(x) = \int_x^1 \left[\int_x^t f'(u) du \right] F_n(x, t) dt. \tag{11}$$

Following Karsli [21], for any $f(t) \in DBV[0, 1]$, we decompose $f(t)$ into four parts as

$$f'(t) = \frac{f'(x+) + f'(x-)}{2} + f'_x(t) + \frac{f'(x+) - f'(x-)}{2} \operatorname{sgn}(t - x) + \delta_x(t) \left[f'(x) - \frac{f'(x+) + f'(x-)}{2} \right],$$

where

$$\delta_x(t) = \begin{cases} 1, & x = t \\ 0, & x \neq t \end{cases}.$$

If we use this equality in (10) and (11), we have the following expressions;

$$\begin{aligned} I_1(x) &= \frac{f'(x+) + f'(x-)}{2} \int_0^x (x - t) F_n(x, t) dt + \int_0^x \left[\int_t^x f'_x(u) du \right] F_n(x, t) dt \\ &\quad - \frac{f'(x+) - f'(x-)}{2} \int_0^x (x - t) F_n(x, t) dt + \left[f'(x) - \frac{f'(x+) + f'(x-)}{2} \right] \int_0^x \left[\int_t^x \delta_x(u) du \right] F_n(x, t) dt. \end{aligned}$$

It is obvious that $\int_t^x \delta_x(u) du = 0$. From the fact that, we get

$$I_1(x) = \left[\frac{f'(x+) + f'(x-)}{2} - \frac{f'(x+) - f'(x-)}{2} \right] \int_0^x (x - t) F_n(x, t) dt + \int_0^x \left[\int_t^x f'_x(u) du \right] F_n(x, t) dt. \tag{12}$$

Using a similar method, for evaluating $I_2(x)$, we find that

$$I_2(x) = \left[\frac{f'(x+) + f'(x-)}{2} - \frac{f'(x+) - f'(x-)}{2} \right] \int_x^1 (t - x) F_n(x, t) dt + \int_x^1 \left[\int_x^t f'_x(u) du \right] F_n(x, t) dt. \tag{13}$$

Combining (12) and (13), we get

$$\begin{aligned} -I_1(x) + I_2(x) &= \frac{f'(x+) + f'(x-)}{2} \int_0^1 (t - x) F_n(x, t) dt + \frac{f'(x+) - f'(x-)}{2} \int_0^1 |t - x| F_n(x, t) dt \\ &\quad - \int_0^x \left[\int_t^x f'_x(u) du \right] F_n(x, t) dt + \int_x^1 \left[\int_x^t f'_x(u) du \right] F_n(x, t) dt. \end{aligned}$$

From the last expression, we can rewrite (9) as follows:

$$I_{n,2}(x) = \frac{f'(x+) + f'(x-)}{2} \int_0^1 (t - x) F_n(x, t) dt + \frac{f'(x+) - f'(x-)}{2} \int_0^1 |t - x| F_n(x, t) dt \tag{14}$$

$$-\int_0^x \left[\int_t^x f'_x(u) du \right] F_n(x, t) dt + \int_x^1 \left[\int_x^t f'_x(u) du \right] F_n(x, t) dt.$$

On the other hand, since

$$\int_0^1 |t - x| F_n(x, t) dt = (D_n |t - x|)(x) \quad \text{and} \quad \int_0^1 (t - x) F_n(x, t) dt = (D_n (t - x))(x),$$

then using these equalities in (14) and taking absolute value, we can re-express (14) as follows;

$$\begin{aligned} |I_{n,2}(x)| &\leq \left| \frac{f'(x+) + f'(x-)}{2} \right| |(D_n (t - x))(x)| + \left| \frac{f'(x+) - f'(x-)}{2} \right| |(D_n |t - x|)(x)| \\ &+ \left| -\int_0^x \left[\int_t^x f'_x(u) du \right] F_n(x, t) dt \right| + \left| \int_x^1 \left[\int_x^t f'_x(u) du \right] F_n(x, t) dt \right|. \end{aligned} \tag{15}$$

According to (6), we write

$$\int_0^x \left[\int_t^x f'_x(u) du \right] F_n(x, t) dt = \int_0^x \left[\int_t^x f'_x(u) du \right] \frac{\partial}{\partial t} \lambda_n(x, t) dt. \tag{16}$$

Using partial integration on the right hand side of (16), we obtain

$$\int_0^x \left[\int_t^x f'_x(u) du \right] \frac{\partial}{\partial t} \lambda_n(x, t) dt = \int_0^x f'_x(t) \lambda_n(x, t) dt.$$

Thus

$$\left| -\int_0^x \left[\int_t^x f'_x(u) du \right] F_n(x, t) dt \right| \leq \int_0^x |f'_x(t)| \lambda_n(x, t) dt$$

and

$$\left| -\int_0^x \left[\int_t^x f'_x(u) du \right] F_n(x, t) dt \right| \leq \int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(t)| \lambda_n(x, t) dt + \int_{x-\frac{x}{\sqrt{n}}}^x |f'_x(t)| \lambda_n(x, t) dt.$$

Since $f'_x(x) = 0$ and $\lambda_n(x, t) \leq 1$, one has

$$\int_{x-\frac{x}{\sqrt{n}}}^x |f'_x(t)| \lambda_n(x, t) dt = \int_{x-\frac{x}{\sqrt{n}}}^x |f'_x(t) - f'_x(x)| \lambda_n(x, t) dt \leq \int_{x-\frac{x}{\sqrt{n}}}^x \bigvee_t^x (f'_x) dt.$$

Make the change of variables $t = x - \frac{x}{u}$, then

$$\int_{x-\frac{x}{\sqrt{n}}}^x \bigvee_t^x (f'_x) dt \leq \bigvee_{x-\frac{x}{\sqrt{n}}}^x (f'_x) \int_{x-\frac{x}{\sqrt{n}}}^x dt.$$

Besides from (6), we can write

$$\int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(t)| \lambda_n(x, t) dt \leq \frac{2nx(1-x) + 2}{n^2} \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x(f'_x) \frac{dt}{(x-t)^2}.$$

Make the change of variables $t = x - \frac{x}{u}$ again, we have

$$\int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x(f'_x) \frac{dt}{(x-t)^2} = \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x(f'_x) \frac{\left(\frac{x}{u^2}\right) du}{\left(-\frac{x}{u}\right)^2} = \frac{1}{x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x(f'_x)$$

and hence, we obtained

$$\left| - \int_0^x \left[\int_t^x f'_x(u) du \right] F_n(x, t) dt \right| \leq \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x(f'_x) + \frac{2nx(1-x) + 2}{n^2x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x(f'_x).$$

Since

$$\frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x(f'_x) \leq \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x(f'_x),$$

it follows that

$$\begin{aligned} \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x(f'_x) + \frac{2nx(1-x) + 2}{n^2x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x(f'_x) &\leq \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x(f'_x) + \frac{2nx(1-x) + 2}{n^2x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x(f'_x) \\ &\leq \frac{2(n+1)}{n^2x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x(f'_x). \end{aligned}$$

Therefore

$$\left| - \int_0^x \left[\int_t^x f'_x(u) du \right] F_n(x, t) dt \right| \leq \frac{2(n+1)}{n^2x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x(f'_x).$$

Using a similar method we have

$$\left| \int_x^1 \left[\int_x^t f'_x(u) du \right] F_n(x, t) dt \right| \leq \frac{2(n+1)}{n^2(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{1-x}{k}}(f'_x).$$

Collecting the estimates, we get (5), i.e.,

$$|I_{n,2}(x)| \leq \left| \frac{f'(x+) - f'(x-)}{2} \right| \sqrt{\frac{2nx(1-x) + 2}{n^2}} + \frac{2(n+1)}{n^2x(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}}(f'_x).$$

In view of c) one has

$$\begin{aligned} |I_{n,1}(x)| &= \left| \int_0^1 [H_n(f(t)) - f(t)] F_n(x, t) dt \right| \leq \int_0^1 |H_n(f(t)) - f(t)| F_n(x, t) dt \\ &\leq \frac{1}{\mu(n)} \end{aligned}$$

holds true. This completes the proof of the theorem.

Proof of Theorem 2. We can write the difference between $(ND_n f)(x)$ and $f(x)$ as a singular Stieltjes integral as follows;

$$\begin{aligned} |(ND_n f)(x) - f(x)| &= \left| \int_0^1 H_n(f(t)) F_n(x, t) dt - f(x) \right| \\ &\leq \int_0^1 |H_n(f(t)) - H_n(f(x))| F_n(x, t) dt + \int_0^1 |H_n(f(x)) - f(x)| F_n(x, t) dt \\ &\leq \int_0^1 \psi(|f(t) - f(x)|) F_n(x, t) dt + \int_0^1 |H_n(f(x)) - f(x)| F_n(x, t) dt \\ &= I_{n,1}(x) + I_{n,2}(x). \end{aligned}$$

Note that for a non-decreasing function ψ

$$-\psi(|f(t) - f(x)|) \leq \psi(|f(t)|) - \psi(|f(x)|)$$

holds true. Since $\psi(|f(t)|) \in DBV[0, 1]$ we can rewrite (9) as follows:

$$\begin{aligned} I_{n,1}(x) &= \int_x^0 [\psi(|f(t)|) - \psi(|f(x)|)] F_n(x, t) dt + \int_1^x [\psi(|f(t)|) - \psi(|f(x)|)] F_n(x, t) dt \\ &= \int_0^x \left[\int_t^x (\psi \circ |f|)'(u) du \right] F_n(x, t) dt + \int_x^1 \left[\int_t^x (\psi \circ |f|)'(u) du \right] F_n(x, t) dt \\ &= I_1(x) - I_2(x), \end{aligned}$$

where

$$I_1(x) = \int_0^x \left[\int_t^x (\psi \circ |f|)'(u) du \right] F_n(x, t) dt \tag{17}$$

and

$$I_2(x) = \int_x^1 \left[\int_x^t (\psi \circ |f|)'(u) du \right] F_n(x, t) dt. \tag{18}$$

For any $(\psi \circ |f|)(t) \in DBV[0, 1]$, we decompose $(\psi \circ |f|)(t)$ into four parts as

$$\begin{aligned} (\psi \circ |f|)'(t) &= \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} + (\psi \circ |f|)'_x(t) \\ &+ \frac{(\psi \circ |f|)'(x+) - (\psi \circ |f|)'(x-)}{2} \operatorname{sgn}(t - x) + \delta_x(t) \left[(\psi \circ |f|)'(x) - \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} \right], \end{aligned}$$

where

$$\delta_x(t) = \begin{cases} 1, & x = t \\ 0, & x \neq t \end{cases}.$$

If we use this equality in (17) and (18), we have the following expressions.

$$\begin{aligned} I_1(x) = & \int_0^x \left\{ \int_t^x \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} + (\psi \circ |f|)'_x(u) \right. \\ & + \frac{(\psi \circ |f|)'(x+) - (\psi \circ |f|)'(x-)}{2} \operatorname{sgn}(u-x) \\ & \left. + \delta_x(u) \left[(\psi \circ |f|)'(x) - \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} \right] du \right\} F_n(x, t) dt \end{aligned}$$

and

$$\begin{aligned} I_2(x) = & \int_x^1 \left\{ \int_x^t \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} + (\psi \circ |f|)'_x(u) \right. \\ & + \frac{(\psi \circ |f|)'(x+) - (\psi \circ |f|)'(x-)}{2} \operatorname{sgn}(u-x) \\ & \left. + \delta_x(u) \left[(\psi \circ |f|)'(x) - \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} \right] du \right\} F_n(x, t) dt. \end{aligned}$$

Firstly, we evaluate $I_1(x)$.

$$\begin{aligned} I_1(x) = & \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} \int_0^x (x-t) F_n(x, t) dt \\ & + \int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u) du \right] F_n(x, t) dt - \frac{(\psi \circ |f|)'(x+) - (\psi \circ |f|)'(x-)}{2} \int_0^x (x-t) F_n(x, t) dt \\ & + \left[(\psi \circ |f|)'(x) - \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} \right] \int_0^x \left[\int_t^x \delta_x(u) du \right] F_n(x, t) dt \end{aligned}$$

It is obvious that $\int_t^x \delta_x(u) du = 0$. From the fact that, we get

$$\begin{aligned} I_1(x) = & \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} \int_0^x (x-t) F_n(x, t) dt + \int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u) du \right] F_n(x, t) dt \\ & - \frac{(\psi \circ |f|)'(x+) - (\psi \circ |f|)'(x-)}{2} \int_0^x (x-t) F_n(x, t) dt. \end{aligned} \tag{19}$$

Using a similar method, for evaluating $I_2(x)$, we find that

$$\begin{aligned}
 I_2(x) &= \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} \int_x^1 (t-x)F_n(x,t)dt + \int_x^1 \left[\int_x^t (\psi \circ |f|)'_x(u) du \right] F_n(x,t)dt \\
 &\quad - \frac{(\psi \circ |f|)'(x+) - (\psi \circ |f|)'(x-)}{2} \int_x^1 (t-x)F_n(x,t)dt.
 \end{aligned} \tag{20}$$

Combining (19) and (20), we get

$$\begin{aligned}
 I_1(x) - I_2(x) &= \frac{(\psi \circ |f|)'(x+) + (\psi \circ |f|)'(x-)}{2} \int_0^1 (x-t)F_n(x,t)dt \\
 &\quad - \frac{(\psi \circ |f|)'(x+) - (\psi \circ |f|)'(x-)}{2} \int_0^1 |t-x|F_n(x,t)dt \\
 &\quad + \int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u) du \right] F_n(x,t)dt - \int_x^1 \left[\int_x^t (\psi \circ |f|)'_x(u) du \right] F_n(x,t)dt.
 \end{aligned}$$

On the other hand, note that

$$\int_0^1 |t-x|F_n(x,t)dt = D_n(|t-x|;x) \leq \sqrt{(D_n(t-x)^2)(x)} = \sqrt{\frac{2nx(1-x)+2}{n^2}}$$

and

$$\int_0^1 (t-x)F_n(x,t)dt = (D_n(t-x))(x) = \frac{1-2x}{n+2}.$$

Using these equalities we can re-expressed $I_1(x) - I_2(x)$ as follows;

$$\begin{aligned}
 I_1(x) - I_2(x) &\leq \frac{(\psi \circ |f|)'(x-) - (\psi \circ |f|)'(x+)}{2} \sqrt{\frac{2nx(1-x)+2}{n^2}} \\
 &\quad + \int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u) du \right] F_n(x,t)dt - \int_x^1 \left[\int_x^t (\psi \circ |f|)'_x(u) du \right] F_n(x,t)dt.
 \end{aligned}$$

Since

$$\int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u) du \right] F_n(x,t)dt = \int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u) du \right] \frac{\partial}{\partial t} \lambda_n(x,t)dt. \tag{21}$$

and using partial integration on the right hand side of (21), we obtain

$$\int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u) du \right] \frac{\partial}{\partial t} \lambda_n(x,t)dt = \int_0^x (\psi \circ |f|)'_x(t) \lambda_n(x,t) dt$$

and

$$\int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u) du \right] \frac{\partial}{\partial t} \lambda_n(x, t) dt = \int_0^{x-\frac{x}{\sqrt{n}}} (\psi \circ |f|)'_x(t) \lambda_n(x, t) dt + \int_{x-\frac{x}{\sqrt{n}}}^x (\psi \circ |f|)'_x(t) \lambda_n(x, t) dt.$$

Since $(\psi \circ |f|)'_x(x) = 0$ and $\lambda_n(x, t) \leq 1$, one has

$$\int_{x-\frac{x}{\sqrt{n}}}^x (\psi \circ |f|)'_x(t) \lambda_n(x, t) dt \leq \int_{x-\frac{x}{\sqrt{n}}}^x \bigvee_t^x (\psi \circ |f|)'_x dt.$$

Make the change of variables $t = x - \frac{x}{u}$, then

$$\int_{x-\frac{x}{\sqrt{n}}}^x \bigvee_t^x (\psi \circ |f|)'_x dt \leq \bigvee_{x-\frac{x}{\sqrt{n}}}^x (\psi \circ |f|)'_x \int_{x-\frac{x}{\sqrt{n}}}^x dt.$$

In view of (6), we can write

$$\int_0^{x-\frac{x}{\sqrt{n}}} (\psi \circ |f|)'_x \lambda_n(x, t) dt \leq \frac{2nx(1-x) + 2}{n^2} \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x (\psi \circ |f|)'_x \frac{dt}{(x-t)^2}.$$

Make the change of variables $t = x - \frac{x}{u}$ again, we have

$$\begin{aligned} \frac{2nx(1-x) + 2}{n^2} \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x (\psi \circ |f|)'_x \frac{dt}{(x-t)^2} &= \frac{2nx(1-x) + 2}{n^2} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x (\psi \circ |f|)'_x \frac{(\frac{x}{u^2}) du}{(-\frac{x}{u})^2} \\ &= \frac{2nx(1-x) + 2}{n^2x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (\psi \circ |f|)'_x \end{aligned}$$

and hence, we obtained

$$\int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u) du \right] \frac{\partial}{\partial t} \lambda_n(x, t) dt \leq \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (\psi \circ |f|)'_x + \frac{2nx(1-x) + 2}{n^2x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (\psi \circ |f|)'_x.$$

Since

$$\frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (f'_x) \leq \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (f'_x),$$

it follows that

$$\begin{aligned} &\frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x (\psi \circ |f|)'_x + \frac{2nx(1-x) + 2}{n^2x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (\psi \circ |f|)'_x \\ &\leq \frac{2x}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (\psi \circ |f|)'_x + \frac{2nx(1-x) + 2}{n^2x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (\psi \circ |f|)'_x \leq \frac{2(n+1)}{n^2x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (\psi \circ |f|)'_x. \end{aligned}$$

Therefore

$$\int_0^x \left[\int_t^x (\psi \circ |f|)'_x(u) du \right] \frac{\partial}{\partial t} \lambda_n(x, t) dt \leq \frac{2(n+1)}{n^2 x} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x (\psi \circ |f|)'_x.$$

By the same way we have

$$- \int_x^1 \left[\int_x^t (\psi \circ |f|)'_x(u) du \right] F_n(x, t) dt \leq \frac{2(n+1)}{n^2(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{1-x}{k}} (\psi \circ |f|)'_x.$$

Thus, one has

$$I_{n,1}(x) \leq \frac{(\psi \circ |f|)'(x-) - (\psi \circ |f|)'(x+)}{2} \sqrt{\frac{2nx(1-x)+2}{n^2}} + \frac{2(n+1)}{n^2 x(1-x)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (\psi \circ |f|)'_x.$$

In view of (c) we obtain

$$I_{n,2}(x) = \int_0^1 |H_n(f(t)) - f(t)| F_n(x, t) dt \leq \frac{1}{\mu(n)}$$

holds for sufficiently large n . Collecting the above estimates the proof of the Theorem 2 is now complete.

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