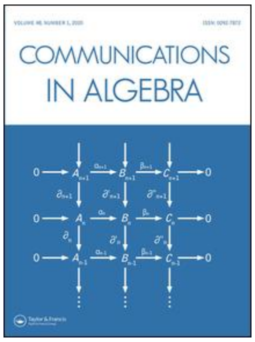

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A CHARACTERIZATION OF RIESZ *n*-MORPHISMS AND APPLICATIONS

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Let $X_1, X_2, \ldots, X_n$ be realcompact spaces and Z be a topological space. Let $\pi : C(X_1) \times C(X_2) \times \cdots \times C(X_n) \to C(Z)$ be a Riesz n-morphism. We show that there exist functions $\sigma_i : Z \to X_i$ $(i = 1, 2, \ldots, n)$ and $w \in C(Z)$ such that

$$\pi(f_1, f_2, \ldots, f_n) = wf_1\circ\sigma_1 f_2\circ\sigma_2 \ldots f_n\circ\sigma_n$$

and $\sigma_1, \sigma_2, \ldots, \sigma_n$ are continuous on $\{z : w(z) \neq 0\}$. This fact extends a result in Boulabiar (2002) and leads to one of the main results in Boulabiar (2004) with a more elementary proof.

Key Words: f-algebras; Realcompact space; Riesz n-morphism.

2000 Mathematics Subject Classification: Primary 46E25, 06F25.

1. INTRODUCTION

For the terminology and notations we refer to Abramovich and Aliprantis (2002), Aliprantis and Burkinshaw (1985), and Boulabiar et al. (2003). The set $C(X)$ of all real-valued continuous functions on a topological space X is a lattice-ordered algebra with respect to the pointwise addition, scalar multiplication, product, and ordering. For each $x \in X$, $\delta_x : C(X) \to \mathbb{R}$ is defined by $\delta_x(f) = f(x)$. Let $\pi : C(X) \to \mathbb{R}$ be an *algebra homomorphism*, that is, π is linear and $\pi(fg) = \pi(f)\pi(g)$ for all f, $g \in C(X)$. Observe that if $f \in C(X)$, then

$$|\pi(f)|^2 = (\pi(f))^2 = \pi(f^2) = \pi(|f|^2) = \pi(|f|)^2,$$

so π is a Riesz homomorphism. Recall that a topological space X is called *realcompact* (see Gillman and Jerison, 1960) if it is homeomorphic to a closed subspace of a product space of real lines. In Ercan (2006) it is noticed that a completely regular space X is realcompact if and only if each (x_α) net converges whenever $(f(x_\alpha))$ converges for each $f \in C(X)$. It is obvious that each compact

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Hausdorff space is realcompact. Let X be a realcompact space. It is well-known that for each algebra homomorphism $\pi : C(X) \to \mathbb{R}$ there exist $r \in \{0, 1\}$ and a unique $x \in X$ such that $\pi = r\delta_x$ (see Shirota, 1952). The well-known proof of this is not given in Zermole–Frankel (ZF) set theory (see Gillman and Jerison, 1960). An elementary proof of this without using the Axiom of Choice can be found in Ercan and Önal (2005). This observation can be applied to generalize Proposition 4.1 of Boulabiar et al. (2003) as follows.

Theorem 1. *Let X be a realcompact space, Y be an arbitrary topological space, and F be a Riesz subspace of $C(Y)$. If $\pi : C(X) \to F$ is a Riesz homomorphism then there exists a function $\sigma : Y \to X$ and $w \in C(Y)$ such that*

$$\pi(f)(y) = w(y)f(\sigma(y))$$

for all $f \in C(X)$ and $y \in Y$ and σ is continuous on $\{y \in Y : w(y) \neq 0\}$.

Recall that the realcompactification υX of the completely regular space X is a realcompact space and that $C(\upsilon X)$ and $C(X)$ are isomorphic as lattice-ordered algebras (see Gillman and Jerison, 1960). From this fact and from the above observation the following well-known theorem can be reproved.

Theorem 2. *Let X and Y be arbitrary topological spaces and $\pi : C(X) \to C(Y)$ be a linear map with $\pi(\mathbf{1}) = \mathbf{1}$. Then π is a Riesz homomorphism if and only if it is an algebra homomorphism.*

The following definition has been introduced and studied in Benamor and Boulabiar (2006).

Definition 3. Let $n \in \{1, 2, \dots\}$ and let $E_1, E_2, \dots, E_n, F$ be Riesz spaces. A map $\pi : E_1 \times E_2 \times \cdots \times E_n \to F$ is called a *Riesz n-morphism* if for each $1 \leq i \leq n$ and $\sigma = (a_1, \dots, a_n) \in E^+$ the following map

$$\pi_{\sigma i} : E_i \to F, \qquad \pi_{\sigma i}(x) = \pi(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$$

is a Riesz homomorphism.

2. A CHARACTERIZATION OF RIESZ *n*-MORPHISM

A representation theorem for Riesz 2-morphism from $C(X) \times C(Y)$ into $C(Z)$ is given in Benamor and Boulabiar (2006) with X, Y compact Hausdorff and Z extremally disconnected. Next, we extend this representation theorem, via an elementary approach, to Riesz n-morphisms from $C(X_1) \times \cdots \times C(X_n)$ into $C(Z)$ with $X_1, \dots, X_n$ realcompact and Z arbitrary.

Theorem 4. *Let $X_1, X_2, \dots, X_n$ be realcompact spaces and Z be a topological space. Let $\pi : C(X_1) \times \cdots \times C(X_n)$ be a Riesz n-morphism. Then there exist functions $\sigma_1 : Z \to X_1, \dots, \sigma_n : Z \to X_n$ and $w \in C(Z)$ such that*

$$\pi(f_1, \dots, f_n)(z) = w(z)f_1(\sigma_1(z))f_2(\sigma_2(z)) \dots f_n(\sigma_n(z))$$

for all $f_1 \in C(X_1), \ldots, f_n \in C(X_n)$, $z \in Z$ *and* $\sigma_1, \ldots, \sigma_n$ *are continuous on* $\{z : w(z) \neq 0\}$.

Before giving the proof we need the following lemma.

Lemma 5. *Let X be a completely regular space and A be a nonvoid subset of* $\{r\delta_x : 0 \leq r \in (0, \infty), x \in X\}$ *which is closed under addition. Then there exists a unique* $x_0 \in X$ *such that*

$$A \subset \{r\delta_{x_0} : 0 \leq r \in \mathbb{R}\}.$$

Proof. The result is clear if A contains only one functional. Otherwise, let $\alpha, \beta \in A$ with $\alpha \neq \beta$. Hence $\alpha = r\delta_x$ and $\beta = s\delta_y$ for some $r, s \in (0, \infty)$ and $x, y \in X$. There exist $t \in (0, \infty)$ and $z \in X$ such that $r\delta_x + s\delta_y = t\delta_z$. Assume by the way of contradiction $x \neq y$. There exists $f \in C(X)$ such that $f(x) = 1$ and $f(y) = f(z) = 0$. It follows quickly $r = 0$ so $\alpha = 0$. Contradiction (recall that $0 \notin A$). □

Now we can give a simple proof of the above theorem.

Proof. We prove the theorem for $n = 2$. The general case can then be deduced easily by induction and a similar argument as used in the case $n = 2$.

Given $z \in Z$, define $\Phi_z : C(X_1) \times C(X_2) \to \mathbb{R}$ by $\Phi_z(f, g) = \pi(f, g)(z)$. It is clear that Φ_z is a Riesz 2-morphism, so $\Phi_z^{\sim}(C(X_1)^+)$ is a subset of the set of Riesz homomorphisms of from $C(X_2)$ into $\mathbb{R}$, where $\Phi_z^{\sim}(f)(h) = \pi(f, h)(z)$. So,

$$\Phi_z^{\sim}(C(X_1)^+) \subset \{r\delta_a : 0 \leq r \in \mathbb{R}, a \in X_2\}.$$

From the above lemma, as $\Phi_z^{\sim}(C(X_1)^+)$ is closed under addition, there exists a unique $\sigma_2(z) \in X_2$ such that

$$\Phi_z^{\sim}(C(X_1)^+) \subset \{r\delta_{\sigma_2(z)} : 0 \leq r \in \mathbb{R}\}.$$

Take $f \in C(X_1)^+$ and observe that there exists unique $0 \leq r(z)_f \in \mathbb{R}$ with

$$\Phi_z(f)(h) = r(z)_f \delta_{\sigma_2(z)}(h) = r(z)_f h(\sigma_2(z))$$

for each $h \in C(X_2)$. Now we have a map $r(z) : C(X_1) \to \mathbb{R}$, $r(z)(f) = r(z)_{f^+} - r(z)_{f^-}$. Since

$$r(z)(f) = \Phi(f, \mathbf{1})(z) \qquad \text{for all } f \in C(X_1)^+,$$

$r(z)$ is a Riesz homomorphism. So there exits a unique $\sigma_1(z) \in X_1$ and $0 \leq w(z) \in \mathbb{R}$ with $r(z)(f) = w(z)f(\sigma_1(z))$. Hence

$$\pi(f, g)(z) = w(z)f(\sigma_1(z))g(\sigma_2(z))$$

for each $z \in Z$, $f \in C(X_1)$ and $g \in C(X_2)$. It is clear that $w = \pi(\mathbf{1}, \mathbf{1})$, and σ_1 and σ_2 are continuous on $\{z : w(z) \neq 0\}$. □

3. APPLICATIONS

In Boulabiar (2004, Theorem 3.1) a representation theorem for d-multiplication in $C(X)$ is given. In fact, the theorem in question includes a misprint, namely, the word "completely regular" should be read "realcompact". Moreover, the (correct) proof of this result is rather complicated. Next, we use our Theorem 4 to get this representation theorem with a much easier approach. To this end, recall that a lattice-ordered algebra A is called a *d-algebra* if $f \geq 0$ and $g \wedge h = 0$ in A imply $(fg) \wedge (fh) = (gf) \wedge (hf) = 0$ (see Boulabiar et al., 2003). Moreover, it is readily verified that $C(X)$ is a d-algebra with respect to a multiplication $\bullet$ if and only if $\bullet$ is a d-multiplication in the sense of Boulabiar (2004). This observation is made in this context to see connection between our next theorem and Theorem 3.1 in Boulabiar (2004).

Theorem 6. *Let X be a completely regular space. Then the following are equivalent*:

(a) *If $C(X)$ is a d-algebra with respect to a multiplication $\bullet$ then there exists a weight function w in $C(X)$, and two functions $\alpha, \beta : X \to X$ such that*

$$f \bullet g(x) = w(x)f(\alpha(x))g(\beta(x))$$

for each $f, g \in C(X)$ and $x \in X$. Moreover, α and β are continuous on $\{x : w(x) \neq 0\}$ and $w = \mathbf{1} \bullet \mathbf{1}$.

(b) *If $C(X)$ is a commutative d-algebra with respect to a multiplication $\bullet$ then there exists a weight function w in $C(X)$, and a function $\sigma : X \to X$ such that*

$$f \bullet g(x) = w(x)f(\sigma(x))g(\sigma(x))$$

for each $f, g \in C(X)$ and $x \in X$. Moreover σ is continuous on $\{x : w(x) \neq 0\}$ where $w = \mathbf{1} \bullet \mathbf{1}$.

(c) *X is realcompact.*

Proof. We prove only (a) $\Longleftrightarrow$ (c). The other implications are similar. Suppose that (c) holds. Let $\bullet$ be a d-multiplication on $C(X)$, so $C(X)$ is a d-algebra with respect to $\bullet$. Then $\pi : C(X) \times C(X) \to C(X)$ be defined by $\pi(f, g) = f \bullet g$ for all $f, g \in C(X)$ is a Riesz 2-morphism. Now we apply Theorem 4 and we get (a).

Now suppose that we have (a) and X is not realcompact. Then there exists a net (x_τ) such that $(f(x_\tau))$ converges for each $f \in C(X)$ but (x_τ) does not converge (see Ercan, 2006). Let $\bullet$ be defined by

$$f \bullet g = (\lim f(x_\tau) \lim g(x_\tau))\mathbf{1}.$$

for all $f, g \in C(X)$. Obviously, $\bullet$ is a d-multiplication and $\mathbf{1} \bullet \mathbf{1} = \mathbf{1}$. Hence, in view of (a), there exist continuous functions $\alpha, \beta : X \to X$ such that

$$f \bullet g(x) = f(\alpha(x))g(\beta(x))$$

for each $f, g \in C(X)$ and $x \in X$. This implies that

$$\lim f(x_\tau) = f \bullet \mathbf{1}(x) = f(\alpha(x))$$

for each $f \in C(X)$. As X is completely regular we have $x_\tau \to \alpha(x)$ (in particular α is constant), this is a contradiction. □

Theorem 3.1 in Boulabiar (2004) should then be stated as follows.

Corollary 7. *Let X be a completely regular space and υX be the realcompactification of X. If $C(X)$ is a d-algebra under the multiplication $\bullet$, then there exist $w \in C(X)$ and $\alpha, \beta : X \to \upsilon X$ such that*

$$f \bullet g(x) = w(x) f^{\upsilon}(\alpha(x)) g^{\upsilon}(\beta(x))$$

for all $f, g \in C(X)$, $x \in X$ and α, β are continuous on $\{x \in X : w(x) \neq 0\}$. Here, $f^{\upsilon} \in C(\upsilon X)$ denotes the unique extension of $f \in C(X)$ and $w = \mathbf{1} \bullet \mathbf{1}$.

The following theorem is known and it is due to Conrad (see Theorem 3.1 in Boulabiar et al., 2003). As an application of the above theorem we can give an alternative proof as follows. Recall first that a lattice-ordered algebra A is said to be an *f-algebra* if $f \geq 0$ and $g \wedge h = 0$ in A imply $(fg) \wedge h = (gf) \wedge h = 0$ (see Boulabiar et al., 2003).

Corollary 8. *Let X be a completely regular space. If $C(X)$ is a f-algebra under the multiplication $\bullet$, then*

$$f \bullet g(x) = w(x) f(x) g(x)$$

for all $x \in X$, $f, g \in C(X)$, where $w = \mathbf{1} \bullet \mathbf{1}$.

Proof. As any Archimedean f-algebra is a commutative d-algebra, from the above corollary, there exits a function $\sigma : X \to \upsilon X$ such that the multiplication can be represented as $f \bullet g(x) = w(x) f^{\upsilon}(\sigma(x)) g^{\upsilon}(\sigma(x))$ and σ is continuous on $\{x : w(x) \neq 0\}$, where $w = \mathbf{1} \bullet \mathbf{1}$. Suppose that $\sigma(x) \neq x$ for some $x \in X$ and $w(x) \neq 0$. Then there exist positive elements $f, g \in C(X)$ such that

$$f^{\upsilon}(\sigma(x)) = g(x) > 0 \quad \text{and} \quad f \wedge g = 0.$$

On the other hand,

$$f \bullet f \wedge g(x) = w(x)(f^{\upsilon})^2(\sigma(x)) \wedge g(x) > 0.$$

This contradiction completes the proof. □

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