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# Change in the Level of Justification in Problem Solving Over Time

# Problem Çözmede Gerekçelendirme Seviyelerinin Zamana Göre Değişimi

Recai AKKUŞ[1]

## Öz

Matematiksel bir bağlamda öğrencilerin eylemleri gerekçelendirme, açıklama, doğrulama gibi bazı ispat şekilleri içermektedir. Her bir ispat formunun öğrencinin matematiksel anlamasına katkı sunan niteliksel seviyesi mevcuttur. Bu çalışmada, 58 matematik öğretmen adayının problem çözme süreçleri gerekçelendirme seviyelerinin zaman içerisindeki değişimini ortaya koymak için analiz edilmiştir. Çalışmanın temel bulguları, zamanla dışsal gerekçelendirme kullanımının azaldığını, içsel kullanımların özellikle ilk beş hafta yükseldiğini ve şematik gerekçelendirmelerin ise dalgalanma yaptığını göstermektedir. Bu bulgular ışığında, öğretmenin dönütleri ve yapılandırılmış yazma sayesinde öğrencilerin zamanla matematiksel gerekçelendirme sunma farkındalıklarının geliştiği söylenebilir.

***Anahtar Kelimeler***: Gerekçelendirme, problem çözme, yazma, matematik eğitimi

## Abstract

Students' actions in a mathematical context contain some sort of proving in the forms of justification, explanation, verification, etc. Each form has levels in terms of the quality that adds to mathematical understanding of students. In this study, 58 teacher candidates' problem solving processes were analyzed across time in terms of the level of justification. The key findings from this study were as the use of external justifications decreased over time, that of internal increased especially for the first five weeks and the use of schematic justifications was fluctuated. The findings suggest that through teacher feedback and structured writing with prompted questions, students can develop an awareness of rigorous mathematical justifications over time.

***Keywords***: Justification, problem solving, writing, mathematics education

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## 1. Introduction

Mathematics educators have focused on proof and justification (i.e., Hanna, 2000) and their importance in elementary and secondary school mathematics (Ellis, 2007; Francisco & Maher, 2005; Maher & Martino, 1996; Reid, 2002; Staples, Bartlo, & Thanheiser, 2012; A. J. Stylianides, 2006) and in college (Harel & Sowder, 1998; 2007; Varghese, 2011). Justification is crucial not only for doing mathematics but also for learning mathematics. Moreover, it is a tool for proving mathematical knowledge. The development of the notion of proof starts at very early ages via argumentation (sometimes, in the forms of explanation, justification, etc.) (A. J. Stylianides, 2006). Therefore, as a form of argumentation justification is a step for proving. Ellis (2007) highlights that students' understanding of mathematical justification at early ages (e.g., in elemantary school) may help them develop more rigorous mathematical thinking (such as proving) at their higher levels in their academic career. The use of justification in this article shall be understood as a chain of arguments that provide reasons for the actions during problem solving. However, it should be noted that justification captures a sort of reasoning structure within an action a person does internally or socially. There are studies focusing on pre-service teachers' challenges in teaching of proving and reasoning (Stylianides, Stylianides, & Shilling-Traina, 2013), abilities of determining whether an argument was a proof (Selden & Selden, 2003). Consequently, research on teachers and teacher candidates is needed to shed light on this issue. In this context, I analyzed the level of justification of elementary mathematics teacher candidates as part of their problem solving processes across time. In addition, this paper speculates that teacher feedback may effect students' justification.

## 2. Theoretical Framework

Knowledge in mathematics becomes mathematical when it is proved. However, an activity in a mathematical context becomes mathematical if one displays a mathematical emotional orientation, which "refers to the implicit criteria members of a community use to decide if explanations are appropriate" (Reid, 2002, p. 24). This notion of mathematics shows that mathematical activity is social in its nature because one has to first ascertain about the truth of his or her assertion and then persuade others about the truth of his or her assertion (Harel and Sowder, 1998; 2007). Ernest (1998) attributed to a number of philosophers such as Manin and Rorty that persuasive nature of proof is what makes it proof. In Manin's words: "A proof becomes a proof after the social act of 'accepting it as a proof.'…" (in Ernest, 1998, p. 183). Hanna (1991) also attributed to Manin by qouting the same phrase to emphasize the fact that the acceptance of a proof is socially agreed among a community of mathematicians. These two actions (ascertaining and persuading) about the truth are not separable but emerges together (Ernest, 1998; Raman, 2003; A. J. Stylianides, 2006). That is, proving is both individual and social such that mathematicians provide mathematical proofs based on their own internal and external arguments with themselves or someone else on the conjectures they formulate by exploring patterns (G. J. Stylianides, 2009). In both contexts, justification (reasoned argument, supported by evidence) is needed. Moreover, Mariotti (2000, pp. 30-31) points out the twofold meaning of proof such that "proving consists in providing both logically enchained arguments … and an argumentation which can remove doubts about the truth of a statement." In other words, proving is not only about the truth of a statement but also about why it is true. Therefore, justification is inherently in proof.

On the other hand, the study of Francisco and Maher (2005, p. 371) "proposes an epistemological distinction between justification and proofs." For them, justification is about students' explanation of their mathematical actions; whereas, "proof is the formal and rigorous argument, which helps mathematicians explain their ideas." However, they strongly emphasize the importance of justification to develop students' mathematical reasoning and understanding in problem solving.

Furthermore, Raman (2003) argued that even though what mathematicians and students do are structurally similar (Weber, 2005), students do not conceptualize that proof is about key ideas (issues of knowledge). Though, students' understanding of importance of proof takes time (Francisco & Maher, 2005). In fact, even high school students heavily rely on examples to justify their reasoning. Ellis (2007, p. 195) therefore argues that "developing students' understanding of justification in middle [and elementary] school may ease the transition to more advanced views of proof in secondary school." In a similar vein, Staples, Bartlo and Thanheiser (2012) demonstrate the importance and practice of justification in classrooms by pointing out the use (purpose) of justification within different communities—the community of classroom and the mathematician community. The crucial importance of proof (and justification) is then the support of mathematical understanding (Hanna, 2000). In the light of the discussion here, we can propose that justification is integral to and a former phase for rigorous proofs.

Students' actions in a mathematical context contain some sort of proving in the forms of justification, explanation, verification, etc. Each form has levels (cognitive stage) in terms of the quality that adds to mathematical understanding of students. Harel and Sowder (1998; 2007), by giving the perspective of subjectivity of proof, describe proof scheme as one's own conception of proof, what counts as truth for a person or a community. The proof schemes, which is based on Harel and Sowder's (1998) work with college mathematics majors, "refer to justifications in general" (p. 275). They also emphasize that their definition of proof schemes was truly based on the subjectivity of proof or justification (Harel and Sowder, 2007).

In this regard, this study aims to add a perspective to the understanding of justification in problem solving by giving the results of teacher candidates' written problem solving processes scaffolded with written prompts. During the teaching of a graduate course, two students came to me and asked for help for finding a subject for their term paper. After a long discussion to understand their interest, I offered them to skim through the problem solving templates that I collected in my undergraduate courses "Problem Solving" and "Problem Solving in Mathematics Education". They copied a bunch of the templates and came back to me by saying that they had realized a change in the teacher candidates' 'reasoning' over time. Therefore, this paper was based on the problem solving templates collected for the purpose of the problem solving courses and the research question was "How does the level of justification in problem solving process change over time?"

## 3. Method

A mixed method approach was adopted for this study because by its nature analyzing problem solving process required a qualitative approach and the results of the qualitative analysis were evaluated using a quantitative perspective. At a university in the West-Blacksea region of Turkey, fifty-eight elementary mathematics teacher candidates (28 second year and 30 third year) who took the elective courses mentioned above participated in this study (see Table 1).

**Table 1. Distribution of the participants by year and gender**

| | Gender | | |
|---|---|---|---|
| Group | Female | Male | Total |
| 2nd Year | 27 | 7 | 34 |
| 3rd Year | 19 | 5 | 24 |
| Total | 46 | 12 | 58 |

**The Courses**

The teacher candidates (hereby I use the word "candidate" to refer to the study group otherwise "student" will indicate the learner in general) in the elementary mathematics teacher education program at the university in which the study was conducted can choose "Problem Solving" in the second year and "Problem Solving in Mathematics Education" in the third year as elective courses, which have been offered for several years at this university. The Problem Solving course includes fundamental concepts in problem solving, problem solving heuristics including Polya's heuristic, cognitive and metacognitive processes in problem solving, and other issues brought by the students. The other course for the third year students, since the Problem Solving course is not a prerequisite, though mostly the students who took the problem solving course prefer to attend to, starts with basic concepts in problem solving and continues with the importance of problem solving in mathematics education, analyzing students' problem solving processes, developing learning activities based on problem solving, and implementation of lesson plans in schools for two weeks. The two courses were taught by the same instructor (the author of this paper) and required the students to actively participate in class argumentation and to work collaboratively on problems. The instructions of the courses were based on the student-centered approach in which students develop their understanding of the topics of the courses based on the ideas come out of the class discussions that ended up with writing assignments.

**Data Collection**

The teacher candidates were taking Analytical Geometry and Analysis courses while taking the problem solving courses within their program. As part of the requirements for the problem solving courses, they solved problems related to the subjects of the week in Analytical Geometry and Analysis courses and returned the solutions in the following problem solving courses. The candidates solved such problems (for ten solid weeks) using a problem solving heuristic developed by the Author (2007). A sample of the problems and questions solved for ten weeks is shown at Table 2. The instructor (the author) had no impact on the candidates' choices of the problems or questions. He was also aware that the candidates might have chosen the problems already solved in the class. However, since the template required students to analyze their solving process, they had to construct their own reasoning in addition to or other than the one taught in the class by the instructors (of the courses Analytical Geometry and Analysis).

**Table 2. A sample of problems/questions solved by a teacher candidate for ten weeks**

| Week | Problem/Question |
|---|---|
| Week1 | Find the equation of the line passing on the point K (1,1) and having a slope of 45 degree with the line 2x+3y-6=0. |
| Week2 | Find a vector which has half-length of the vector (16,-12) and opposite direction. |
| Week3 | What is the area of the square whose opposite sides are on the line 3x+4y-12=0 and its symmetry line about the point A (1,2). |
| Week4 | Find the closest point of the circle $x^2 + y^2$ -4x-2y-20=0 to the point (10,7). |
| Week5 | Find the circles with 10 units diameter that are tangent to the line 3x-4y-13=0 at the point (7,2). |
| Week6 | The parabola $y^2$ = 4cx passes through the point (-8,4). Find its tangent line that is parallel to the line 3x+2y-6=0. |
| Week7 | Find a point M on the ellipse $x^2/25 + y^2/16=1$ that satisfies \|MF\|=4; F is the focal point. |
| Week8 | Find the equation of the hyperbole that passes through the foci of the ellipse $x^2/25 + y^2/9=1$ and that has the focal points A and A`. |
| Week9 | Find the equations of the focal rays at the point (2, -5/3) of the ellipse $x^2/9 + y^2/5=1$. |
| Week10 | Make the circle $x^2 + y^2$ -4x+8y+15=0 central. Find the new coordinates of the point A(4,-3) on the circle after this translation. |

The purpose of the template is to support students' problem solving processes through a series of promoting questions (see Table 3). The template allows students to evaluate their own ideas via writing (Kenyon, 1989). The template was not designed to help students improve their proving abilities per se, rather was developed to require students to *justify* their problem solving processes. The problem solving template is used in three phases. First, students solve the problem via the promoting questions 1 to 6, then share their solutions and ideas with their friends, and finally fill out 7th and 8th questions in the template. Consequently, the teacher candidates formed groups of two or three and worked with the partner(s) throughout the semester. The candidates, especially before the feedback, justified their problem solving methods and solutions by simply saying such as "*My solution is true because I used the formula*." Especially for the first three weeks, the instructor gave written feedback for each template of all the candidates. He also clarified the expectations from each question in the template overall in the class (e.g., *you have to give a mathematical reasoning why the formula works in this problem; you have to make an evaluation for your learning mathematics or for your problem solving procedure*). One of the main purposes of feedback was to create classroom norms that meet course requirements and the instructor's expectations, and to guide students in the use of the problem solving template.

**Table 3. Problem solving template**

| Question/Problem: | | |
|---|---|---|
| LET'S FIRST UNDERSTAND THE PROBLEM! | | |
| 1. What is given? What is the relationship among givens? What are the conditions of the problem? | | |
| 2. What does the problem want me to find? What is its relation with givens? Can I write a mathematical statement? | | |
| 3. How can I draw a table or diagram according to the problem? | | |
| 4. How can I solve the problem? | | |
| 5. What did I do? What did I think? What did I struggle with? | | |
| 6. What are my reasons and justifications? | | |
| a) How do I know the process in the solution is correct? | b) Is my solution reasonable? Why? | c) Are there other ways to solve the problem? |
| 7. What do the others say?<br>• What are the similarities and differences when I compare my solutions and ideas with my friends? | | |
| 8. Evaluation – How have my ideas changed? Why? Or Why not? What did I learn from this problem? What is my mathematical conclusion? | | |

In order to better represent the results of this study, I explained the problem solving template in terms of the social and individual aspects of knowledge construction. First of all, in mathematics one *proves* to *ascertain* for oneself or/and *persuade* others about the truth asserted by the individual (Harel & Sowder, 1998). They have to give an outline of the solution for the problem (which includes any sort of justification), they have to be certain about their solution (reason their solutions for themselves) and explain it to others (convince their partners). The problem solving template itself is a scaffold for students' reasoning, which students do rarely deploy when working by themselves.

### Analysis

Data analyses were run in two rounds. The first round passed through three phases. First of all, randomly chosen five problem solving templates were analyzed in terms of justification and reasoning and a coding system was developed by independent researchers (two research assistants and one assistant professor). Secondly, codes were compared; there was an 89% consistency and discrepancies were resolved. The teacher candidates' justifications were categorized according to commonalities in justifications (see Table 4).

**Table 4. Explanation for coding and themes**

| Themes | Codes | Explanation |
|---|---|---|
| External Justification | Previous Knowledge | Justifying through previous knowledge without explanation |
| | Friends | Justifying own solution based on friend's solution without reasoning |
| | Computational Justification | Justifying based on the computational correctness within the solution |
| | Theory-Formula-Rule | Referring to any theory, formula and/or rule without reasoning or explanation |
| Schematic Justification | Refer to Previous Justification | Referring to previous justification within the solution |
| | Backward Controlling | Justifying with backward checking for the solution/computations |
| Internal Justification | Search for Other methods | Realization of other solution methods and search for them |
| | Mathematical and/or Subjective evaluation | Evaluation of the problem solution with a mathematical point of view by making connections among related subjects and self-evaluation |

The common themes come out of the analysis (Table 5) of data looked like Harel and Sowder's proof schemes (external, empirical, and analytical), which are more rigorous on proving, yet they did not fit into the schemes. I preferred to use justification rather than proof as it is more appropriate for the process of problem solving, which involves any combination of various arguments. The candidates who used *external justification* often referred to their old experiences about the type of the problem or to the general solution methods learned in the "class".

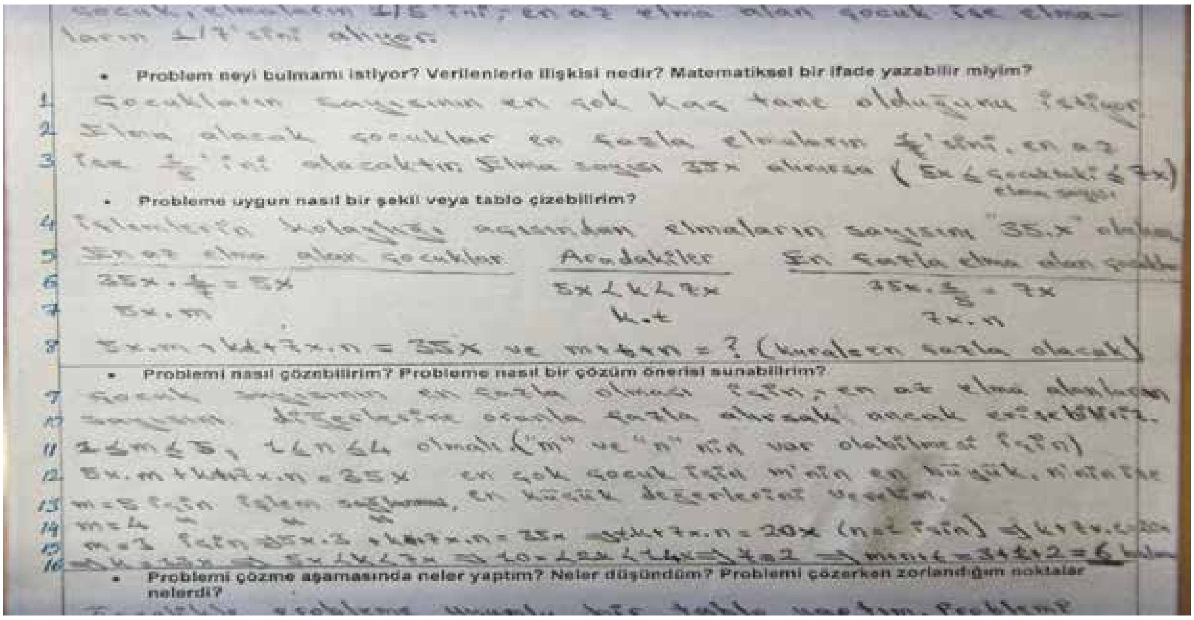



**Figure 1. The solution of the problem (an example for internal justification)**

Therefore, they did not justify their solution further. The ones who used *schematic justifications* mostly mentioned that they developed a method over time for the certain type of problems therefore they intuitively knew it would work or they referred to the basic structure of the solution by linking specific step in the solution (Mayer, 1982). The visible distinction between external and schematic justifications is that in the latter the teacher candidates mostly referred to their own developed methods and to the data in that specific problem. The candidates who used *internal justification* elaborated their reasoning with mathematical facts and definitions, and conceptualize the knowledge come out as a result of the solution. Therefore, it is more about self-awareness of problem solving process.

A teacher candidate solved the following problem: “Some apples are being shared among the kids. The child who takes the most number of apples takes 1/5 of the apples and the child who takes the least number of apples takes 1/7 of the apples. How many kids are there at most?” He used internal justification when solving the problem (see Figure 1). His explanation was like this: Let’s take the number of apples 35x for the easiness of computations (line 4)…. For the number of kids to be the most, we must think that there are more kids taking the least number of apples compared to the others (lines 9-10). Using this reasoning and the table in lines 5-7, he proposed that $1 \leq m \leq 5$ and $1 \leq n \leq 4$ and we must enter the highest value of m and the smallest value of n into the equation (lines 12-13). He therefore eliminated 5 and 4 for m in lines 13-14 and decided m must be 3 (yet he did not justify this reasoning). As we see in this example, the teacher candidate, using his previous experience (lines 3-4), decided to take the number of apples 35x for the easiness of computations. Even though he later solved the problem with specific examples, he had set off sub goals for the solution, which indicates his awareness of the problem solving process. There were cases where we hesitated to decide the type of justification. We solved such cases by looking at the solution as a whole unit and making “the best judgment” together (Yopp & Ely, 2016)

**Table 5. An example of analysis.**

| Type of Justification | Examples | Corresponding Coding |
|---|---|---|
| External Justification | Because I have learned how to solve this kind of problems and I followed the steps.<br><br>I understood that I was correctly solving the problem by looking at the book.<br><br>The formulas direct me to the right operations.<br><br>This type of problems are generally solved in one way. Since we were taught in this way, and we haven’t got enough knowledge about this subject, we couldn’t propose an alternative solution [after a discussion with the partner]. | Previous knowledge, referring to the book or a friend |
| Schematic Justification | I used the methods I previously developed for this type of problem.<br><br>I mentally imagined the figure while solving the problem even though the shape doesn’t look like it.<br><br>My solution is reasonable because I used all the givens in the problems and did not make any error.<br><br>If we look at the graph, we can see that the closest point passes through the center of the circle. | Previous developed method or the data in the problem |
| Internal Justification | I truly understand why the axes were named with these letters. For example, considering k-axis, if a vector like ϑ= ai+bj+ck is given, I now know that i represents (1,0,0); j represents (0,1,0); k represents (0,0,1) and that the equivalent of the vector would be a(1,0,0)+b(0,1,0)+c(0,0,1) that equals to (a,b,c). Even we can conceptualize our physics knowledge through reasoning from this problem.<br><br>Is my starting point reasonable? If you don’t take the three points on the same line, M, A, C compose a triangular area. Then, the triangle inequality must be hold; $\|MC\| - \|MA\|\| \leq \|AC\| \leq \|MC\| + \|MA\|$. If we want the shortest distance, $\|MC\| + \|MA\|$ must be equal to $\|AC\|$. This, then, makes the three points not triangular but linear. | Mathematical support and deductive conclusion |

In the third phase of the first round, each theme appeared in the templates was counted separately for every week. To illustrate, the theme external justification appeared in a candidate's first week problem solving template, let us say, two times; therefore, that teacher candidate had 2 points for the external justification in the week 1. Then, all the candidates' points for external justification were summed up to get the total points of the first week. Based on the total points for each week, a repeated measurement model (dependent t-test) for related groups was run to investigate the growth of each justification type and possible comparisons among the weeks. Based on the results of the first round, another round of analysis was run to investigate whether the teacher candidates developed their internal justifications based on schematic ones, and, if so, how the transition occurred. For this analysis, the templates which contained at least two of the themes (i.e., schematic and internal) were chosen. Of those, the three groups' templates (total nine templates) were analyzed in detail.

## 4. Results and Discussion

The results indicated that the teacher candidates' justifications showed variations across ten weeks (Table 6). Pairwise comparisons of the weeks for each justification type yielded that the uses of the external justifications during the first weeks (especially the second week) were (significantly) higher than the uses during the later weeks (especially, the 6$^{th}$ , 8$^{th}$ and 10$^{th}$ weeks). Furthermore, the tenth week's result was significantly lower than that of the second, seventh and ninth weeks. While the only significant differences for the schematic justifications were between the 4$^{th}$ and 5$^{th}$ and the 4$^{th}$ and 7$^{th}$ weeks, there were many significant difference values for the internal justifications. For example, the first week's results were significantly lower than that of the 4$^{th}$ , 5$^{th}$ , 7$^{th}$, 8$^{th}$, and 9$^{th}$ weeks. Moreover, the second week's results were significantly lower than that of the 5$^{th}$, 8$^{th}$ and 9$^{th}$ weeks. Only backwards drop in the results for the internal justifications was between the 5$^{th}$ and 6$^{th}$ weeks. The use of the internal justifications in the 6$^{th}$ week was significantly lower than that in the 5$^{th}$ week. The results increased during the later weeks. Even though in the tenth week the use of the internal justifications decreased, this drop was not significant nor was it lower than that in the first weeks. This statistical analysis indicated that as the use of the external justifications decreased over time, the use of the internal justifications increased especially for the first five weeks due to writing and teacher feedback. The later drop in the usage of the internal justifications might be related to the candidates' exhaustion using the template during the final week of the courses (e.g., *We learn a lot using the templates but it takes time to fill out the form. We have exams next week…*-Personal communication with the teacher candidates in the class).

The candidates tried to justify their problem solutions by mostly referring to their old experiences and to memorized rules. The candidates at this level tended to strictly follow the rules. Through writing and probable teacher feedback, they realized that they had been solving the problems by memorization and questioned the problem solving methods they had been using: "Why do we use the same formula all the time?" "We do the problems by memorization!" "Where does this rule come from?" This kind of questioning led the teacher candidates to think deeply on the problems. Therefore, the candidates wanted to find different solutions for the problem and chose, instead of memorizing the rules, to understand the logic behind the rules and formulas by questioning their earlier knowledge units: "If $c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} = \mathbf{0}$ , then we called linear dependent. But why?". For example, a teacher candidate wrote: "I generally make little explanation in problem solving. I obsessed with a method that leads me to the solution when drawing or thinking about the givens. Even though I fill out the template while solving the problem, I have hard times to write down my ideas. But as I used the template, I could easily put my ideas on the paper. I am used not to think on the structure of the formulas; but now, as I focus on this [filling out the template], I realized that my solution is more reasonable." This caused that the candidates tended not to use external justifications, rather constructed their own justifications by developing mathematical reasoning.

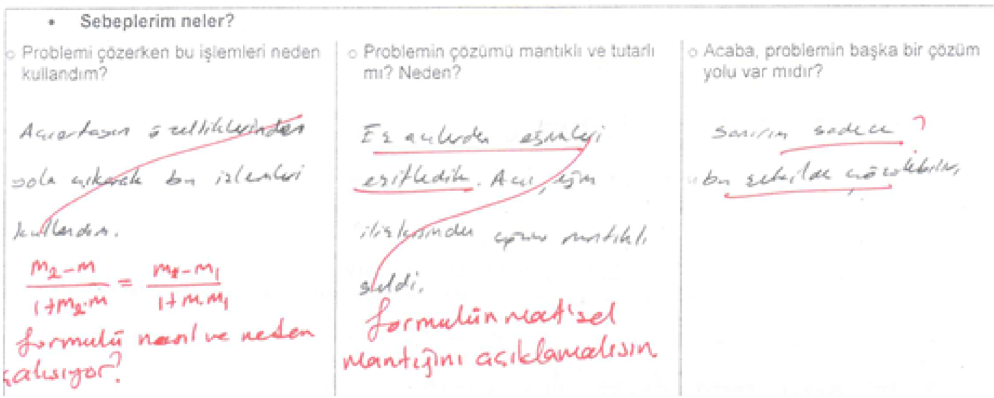
The candidates used schematic justifications for almost all solutions. The nature of schematic justifications let students concretely see their assumptions because they can try and make sure that their solution methods work well (Sandefur et al., 2013). Here is an example that depicts the candidates' reasoning was schematic: *"To understand that my solution is reasonable, let's try specific values. Given the equation of the line AB, enter the value of middle point. It works. … since all the points satisfied the rule, my solution is reasonable."* Even though empirical testing is a part of proving process, if the purpose is not to understand why the formula works, this may be an indication of lack of understanding of proving. However, there were teacher candidates who developed this awareness especially after the teacher feedback. For example, before the feedback, a teacher candidate wrote "*I learned a practical way for solving this problem. Instead of finding the second equation, we could have first found the distance of the point to the line and calculated the area of the square on that line."* After the feedback (4$^{th}$ and 5$^{th}$ templates), she wrote "*I saw that I could easily solve this problem; but I also realized that I used different methods other than the memorized ones. I had never* *thought of why the center of circle was so important for this type of problems.*" The figures 2a and 2b below show how the candidates improved their reasoning based on the teacher's feedback (the two templates belong to the same candidate).

**Table 6. Pairwise comparison table across the weeks ($Week_i$ – $Week_j$)**

| | | $Week_j$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $Week_i$ | Scheme | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | Ext. | -.19* | .01 | -.01 | -.07 | .06 | -.07 | .05 | -.10 | .13 |
| | Sch. | .04 | .04 | -.14 | .08 | -.11 | .06 | .04 | -.06 | -.11 |
| | Int. | -.06 | -.14 | **-.22** | **-.37** | -.09 | **-.26** | **-.25** | **-.29** | -.14 |
| 2 | Ext. | | .19 | .18 | .11 | **.25** | .12 | **.24** | .09 | **.32** |
| | Sch. | | .00 | -.18 | .04 | -.16 | .01 | .00 | -.10 | -.16 |
| | Int. | | -.09 | -.17 | **-.32** | -.03 | -.20 | **-.19** | **-.23** | -.08 |
| 3 | Ext. | | | -.02 | -.08 | .05 | -.08 | .04 | -.11 | .12 |
| | Sch. | | | -.18 | .04 | -.16 | .01 | .00 | -.10 | -.16 |
| | Int. | | | -.08 | **-.23** | .06 | -.11 | -.11 | -.14 | .01 |
| 4 | Ext. | | | | -.06 | .07 | -.06 | .07 | -.09 | .14 |
| | Sch. | | | | **.22** | .03 | **.20** | .18 | .08 | .03 |
| | Int. | | | | -.15 | .14 | -.04 | -.03 | -.07 | .09 |
| 5 | Ext. | | | | | .14 | .01 | .13 | -.03 | .21 |
| | Sch. | | | | | -.19 | -.02 | -.04 | -.14 | -.19 |
| | Int. | | | | | **.28** | .11 | .12 | .08 | .24 |
| 6 | Ext. | | | | | | -.13 | -.01 | -.16 | .07 |
| | Sch. | | | | | | .17 | .16 | .05 | .00 |
| | Int. | | | | | | -.17 | -.17 | -.20 | -.05 |
| 7 | Ext. | | | | | | | .12 | -.03 | **.20** |
| | Sch. | | | | | | | -.01 | -.12 | -.17 |
| | Int. | | | | | | | .01 | -.03 | .12 |
| 8 | Ext. | | | | | | | | -.15 | .08 |
| | Sch. | | | | | | | | -.10 | -.16 |
| | Int. | | | | | | | | -.04 | .12 |
| 9 | Ext. | | | | | | | | | **.23** |
| | Sch. | | | | | | | | | -.05 |
| | Int. | | | | | | | | | .15 |

***** Mean difference (e.g., $Week_1$ – $Week_2$)
**Bold** represents the mean difference is significant (α = 0.05)
Ext.: External; Sch.: Schematic; Int.: Internal

Before the feedback (3rd temp.)

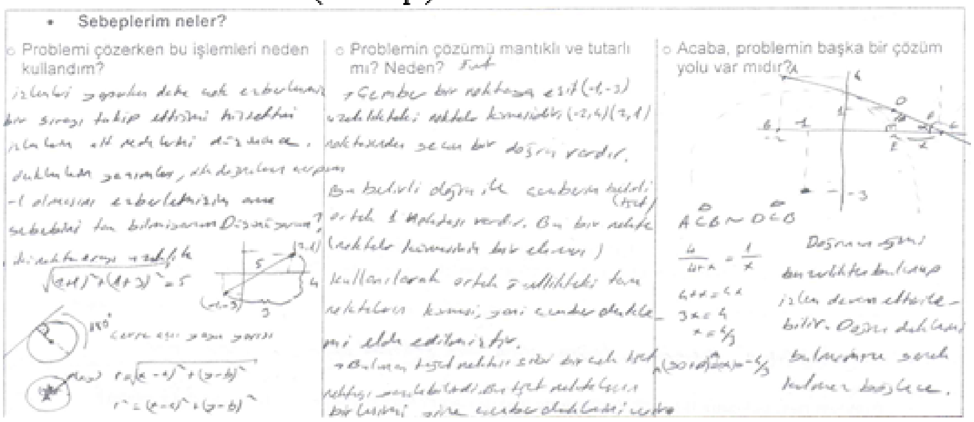



Translation:

| | | |
|---|---|---|
| S: I used the properties of angle bisector.<br>T: How and why does the Formula work? | S: Equal angles, equal slopes. The solution is reasonable because of angle and slope relation.<br>T: You have to explain the mathematical logic behind the Formula. | S: I think this is the only way to solve it.<br>T: ? |

**Figure 2a. An example of a candidate's reasoning before the feedback**

After the feedback (5th temp.)

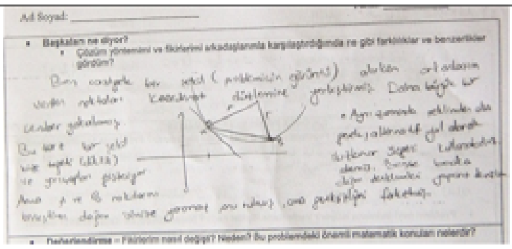



Translation:

| | | |
|---|---|---|
| I realized that I had memorized the rules. Once thinking about how the rules worked, I had memorized that the multiplication of the slopes of the perpendecular lines is -1, but I don't know why. I am thinking?<br>The distance between two points: (equalities and graph)<br>(tangent line and circle)<br>(the formula of circle) | Yes. A circle is the set of all points that are equidistant from a given point. There is a line passing through the points (-2,4), (2,1). There is a certain common point of this line and the circle. This common point and all similar points were used to form a circle. I might have found several tangent points like the one I found. These tangent points again satisfy the circle equation. | (graph)<br><br>The slope of the line could have been found using the similarity of triangles. Therefore, we did not need to find the equation of the line. |

**Figure 2b. An example of a candidate's reasoning after the feedback**

Most of the candidates explained their reasoning, before the feedback, by saying "*I didn't do any calculation errors, so my solution is reasonable.*" Such a reasoning like in Figure 2a, was questioned by the teacher by asking further questions: *How and why does the formula work? You have to give a mathematical logic for your solution*. This helped the candidate understand how she would support her ideas without restating (Zaslavsky & Shir, 2005) what she had done (Figure 2b). Therefore, this was a jump for understanding that in justification why the solution is true is as important as whether it is true. Another teacher candidate made the following evaluation: *Even though my ideas haven't changed, I grabbed the logic behind the concepts. ... Finding the correct result may not show that we understood the logic of the* *problem. Memorized rules have to be questioned* (Template 8).



Translation:

- What do the others say?

While I drew a random figure for the problem, my friend located the points given on the coordinate system. She had a bigger circle which helped us to see the tangent (perpendecular) and radius. Also, she offered an alternative solution using the figure; the isosceles triangle.

**Figure 3. Importance of drawing.**

The teacher candidates who used figure and graphic to justify their solutions emphasized the importance of visual representation in problem solving. "*I couldn't draw the graph, so I don't know if it is reasonable.*" "*I drew a picture in order to reduce the error. I construct a series using the pattern in the drawing.*" A teacher candidate realized the importance of drawing after a discussion with the partner. She pointed out that her partner's graph gave her an idea about an alternative solution which she could not see from her own drawing (see Figure 3).

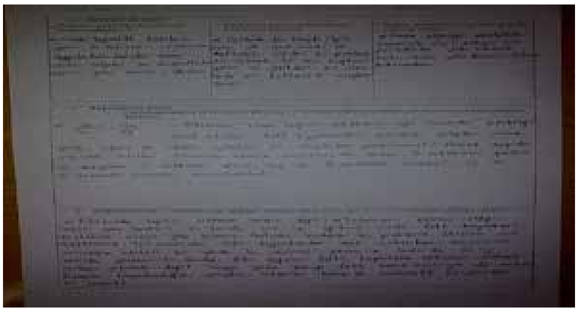
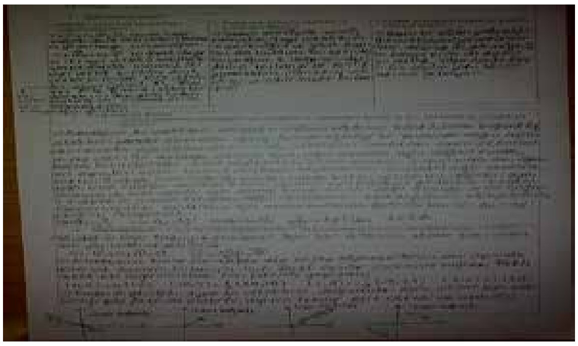
The pairwise comparisons across the weeks for schematic and internal justifications illustrate that in the 5$^{th}$ and 6$^{th}$ weeks there were rapid fluctuations that needed special attention (see Table 6). For example, in the fifth week, the use of schematic justifications was very low and the use of internal justifications was very high; and other way around in the sixth week compared to the rest of the weeks. The types of questions for these two weeks were similar (e.g., one group solved a problem about a circle tangent to two lines in the fifth week, another group solved the same or similar problem in the sixth week). This means that it was not related to the task. One possible explanation for this case was that the fifth and sixth weeks were the mid-term weeks at the university. Some candidates might have put less emphasis on the problems during the week of the mid-terms because they sometimes complained about filling out the template and having a break for a week. The mean differences between consecutive weeks for internal justifications were in favor of the latter weeks except for the sixth and tenth weeks. In other words, there are only two backdrops in internal justifications (from fifth to sixth and from ninth to tenth). These two time periods were the exam weeks. Even though the 10$^{th}$ week's results got lower compared to the 9$^{th}$ week's, it was not less than that of the first two weeks. Moreover, the 9$^{th}$ week's results were higher than that of all the other weeks (except for the 5$^{th}$ week).

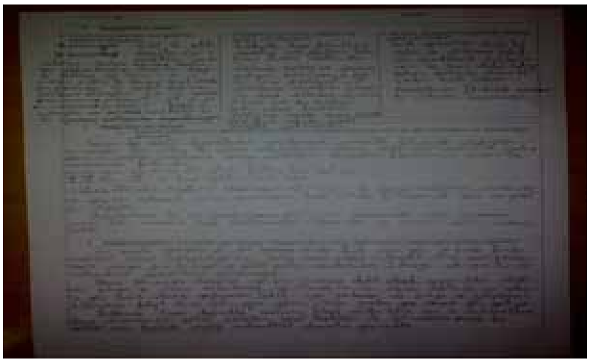
Most of the candidates used both schematic and internal justifications; yet it seems that the ones who used internal justifications considered giving specific examples after justifying their solutions analytically. The following question was individually solved by three teacher candidates in the same group in the fifth week, after the feedback. "If the vectors $\vec{u} = (k-1,3)$, $\vec{v} = (1, k+1)$ are linear dependent, *k*=?" The three candidates analyzed the givens in terms of the definition of linear dependency in algebra, yet while the two of them (OF and HK; the candidates' initials) used the definition introduced by their teacher in the classroom (*Given n vectors* $\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}, \ldots \overrightarrow{u_n}$ *and* $c_1, c_2, c_3, \ldots c_n$ *real numbers at least one of which is different from zero; if* $c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} + \cdots + c_n\overrightarrow{u_n} = \mathbf{0}$ *, then these vectors are linear dependent.*), the third one (TK) inferred that "*if these two vectors are linear dependent, that means they are parallel*." He later concluded during the planning phase that "*if they are parallel, their coordinates must be proportional, that is*

$$\frac{u_x}{v_x} = \frac{u_y}{v_y}$$

When I analyzed their reasoning about the methods they had chosen, I found three different ways of justification (see Table 6). OF stated that he used the linear dependency rule, which is why it was true, without further explanation. On the other hand, even though HK wrote that she used the linear dependency definition given in the classroom, she further scrutinized why the rule works. TK, who inferred linear dependency as parallelism, however made a connection between vectors and lines and justified his method using the parallel lines. Therefore, here we can see that the same reasoning can be justified in different ways: externally (OF), empirically (HK) and internally (TK). Furthermore, while OF evaluated the group discussion in terms of the method they used, HK and TK focused on the concept of linear dependency by comparing different solutions to improve their understanding further. For example, HK stated that her friend's (TK) saying that it was related to parallelism, she combined her own method and TK's method, and concluded that linear dependency has different forms as depicted in the Figure 4. Yet, she wanted to exemplify by giving specific values to make sure it worked. When I analyzed HK's earlier templates, she used both types of justifications but less emphasis on the need to learn about justification of solutions mathematically. This case is an example of what A. J. Stylianides (2009) argues "learning about more secure methods for validating patterns." She not only translated the problem into mathematical language but also questioned her method. She moved among several mathematical transformations to justify the solution. However, as mentioned, she was still within the boundaries of the teacher, classroom work. Harel and Sowder (1998) attributed to Fischbein and Kedem (1982) that students may want to exemplify by specific values to check whether their method or solution is true even though they provide rigorous proof. They stated that this is an indication of uncertainty about their proof. For this reason, HK's solution can be considered as schematic (yet to some extent it might have been internal). On the other hand, TK demonstrated more rigorous thinking by making connections between different knowledge units within mathematics. He also developed a better understanding of justification, independent of external sources: "*Yet, I always prefer to use the ones to which I connect my experiences.*" This shows his awareness of knowledge construction.

**Table 6. OF's, HK's and TK's reasonings about the problem given above.**

| Original examples in the native language | Translation into English | | |
|---|---|---|---|
| | a) Why did I use the operations in the solution? | b) Is my solution reasonable? Why? | c) Are there other ways to solve the problem? |
|  OF | I applied the linear dependency rules, $c_1.u_1+c_2.u_2\ldots c_n.u_n=0$. I used the givens in the question in this direction as the solution method. | There were many questions in the notebook related to this problem and I kind a translated them into this problem for a control. | Since the term linear is related to parallelism concept, I tried to solve using parallelism but couldn't. |
|  HK | If $c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} = 0$, then we called linear dependent. But why? I will try to show as much as I can. $c_1\overrightarrow{u_1} = -c_2\overrightarrow{u_2}$, this means that when I multiply the vectors with two constants the new vectors are equivalent but opposite. (*I saw not always opposite*) Because, $\vec{A} = -\vec{B}$ means that $\vec{A}$ *and* $\vec{B}$ are opposite vectors. $c_1$ or $c_2$ cannot be zero, neither can both. | I didn't limit myself with the givens. I found the constants and which vectors can be formed. I did not see any problem with the k's I found in this way. | I don't think there is another solution. We have seen this kind of problem in the linear algebra course, we solved exactly like this. No other solution comes to my mind. |

| | | | |
|---|---|---|---|
|  TK | Let's think two vectors parallel. If it is given that they are parallel, there is no intersection. If the two vectors do not intersect, the ratios of their x's and y's are equal. Let's think about the two parallel lines. The coefficients of x and y in the equations must be equal. This is because of the equity of the slopes. | Remembering that linear dependency is equivalent of parallelism was the break point. The logic of the solution can be controlled by replacing the value of k; by finding the real values of the vectors; by checking whether the slopes of the vectors are equal. The slopes must be equal because there exists parallelism. | There is also the way we have seen in the classroom. Yet, I always prefer to use the ones to which I connect my experiences. I am sure we will talk about this. |

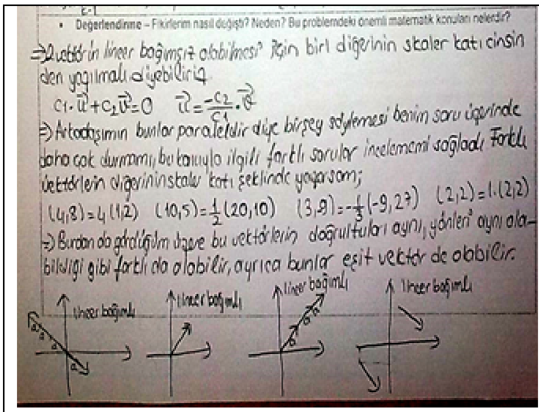



**Figure 4. HK's evaluation**

Another group of candidates (HT, PTA, CT) worked on a problem of finding the equation of the circle passed through the three points given (8th week's problem). Individual work showed that the candidates solved the problem using generic forms of circle equation, which is, $x^2 + y^2 + Dx + Ey + F = 0$ or $(x-a)^2 + (y-b)^2 = r^2$, where $a$ and $b$ are the coordinates of the center. They solved the equation system they obtained. However, after group discussion, they realized that they needed to justify why the formula made sense to them. One candidate (HT) wrote, "*I followed a classic way to solve the problem. I noticed that I could not justify my solution as mathematical as my friend. She drew segments between the three points, added perpendicular lines from the center of the circle to the segments, and justified her solution by using the properties of isosceles triangle. Which perfectly makes sense why the three points meet at the center* [have the same distance]." This teacher candidate noticed that entering specific values into the formula was not enough to justify the reasoning. Therefore, she developed her understanding of mathematical justification. This was a transition from schematic to internal justification. Even though her own solution was based on examples, after the evaluation she made connection to internal justification. Indeed, she (HT) showed this change in the ninth week's problem, which was "finding the circles of a given radius that are tangent to a line at a given point." After understanding the problem (e.g., identifying the givens and the conditions of the problem and drawing an appropriate figure), she proposed a solution method by setting sub-goals for herself. "*When I first drew the figure, I realized that I can obtain equations using the formula of the (perpendicular) distance of a point to a line (since the formula contains absolute value, I will get two different equations, which means there will be two circles).*" It seems that sharing and discussing problem solutions helped the teacher candidates reflect on their overall understanding of justification and develop rigorous mathematical explanations.

## 5. Conclusion

This study is important in terms of the development in students' understanding of mathematical justifications even though most of the problem solving and classroom activities do not directly aim to scaffold such understanding. I speculate that this understanding developed through problem solving activities to be written using problem solving template which requires any sort of justifications in the process. Therefore, this study highlights that through writing structured with prompted questions and teacher feedback, students not only can improve their understanding of mathematics but also develop an awareness of mathematical knowledge construction. In the study reported here, the candidates had experiences that required them to justify every bit of their actions in problem solving. As a result of developing more sophisticated strategies for justification, their use of external justifications decreased whereas internal ones increased to some extent. The prompted questions in the problem solving template helped the candidates reflect on their mathematical actions in and on time. Furthermore, the structure of the template pushes students to justify their solutions not only for themselves but also for others. This feature helps students not only see the social and individual sides of justifying but also experience the processes of advanced mathematical thinking (convince yourself, convince a friend, and convince an enemy) as Mason, Burton and Stacey (1982) described. Moreover, it can be speculated that teacher feedback creates norms for justifications in problem solving and for the expectations of teacher (playing a role of enemy or advocate of the devil).

The study also, at some level, speculates that students might use schematic justifications as a foregoing step for internal justifications with the limitation of when and how in time the transition may occur. I think that a scaffold by a teacher or an expert in time might help students develop the links between schematic and internal justifications such that when it is reached to axiomatic explanation there is no need to give empirical *evidence* (as in HK's case) to justify; empirical data can be used to convince others but not to justify the solution. However, Healy & Hoyles (2000) found that students presented examples as evidence to convince and explained in words by supporting with pictures. They argued that "empirical data convince whereas words and pictures, but not algebra, explain" (p. 415). The question here that needs to be asked is whether students try to convince themselves or others. Yet, this idea needs for further research by designing structured scaffold aiming at justifying or even proving.

## 6. Suggestions

In the light of these findings, writing can be a part of problem solving in mathematics lessons. Mathematics teacher candidates can discover their own problem solving behaviors that will help them to follow their future students' problem solving procedures. As justification is an important process in learning mathematics and discovering mathematical structures, teachers can use the template to foster students' thinking.

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