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ON THE COHOMOLOGY RING OF THE INFINITE FLAG MANIFOLD **LG/T**

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Abstract

In this work, we discuss the calculation of cohomology rings of LG/T. First we describe the root system and Weyl group of LG, then we give some homotopy equivalences on the loop groups and homogeneous spaces, and investigate the cohomology ring structures of LSU_2/T and ΩSU_2. Also we prove that BGG-type operators correspond to partial derivation operators on the divided power algebras.

1. Introduction

In [10], Kumar described the Schubert classes which are the dual to the closures of the Bruhat cells in the flag varieties of the Kač-Moody groups associated to the infinite dimensional Kač-Moody algebras. These classes are indexed by affine Weyl groups and can be choosen as elements of integral cohomologies of the homogeneous space $\widehat{L}_{\mathrm{pol}}G_{\mathbb{C}}/\widehat{B}$ for any compact simply connected semi-simple Lie group G. Later, S. Kumar and B. Kostant gave explicit cup product formulas of these classes in the cohomology algebras by using the relation between the invariant-theoretic relative Lie algebra cohomology theory (using the representation module of the nilpotent part) with the purely nil-Hecke

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rings [9]. These explicit product formulas involve some BGG-type operators A^i and reflections. Using some homotopy equivalances, we determine cohomology ring structures of LG/T where LG is the smooth loop space on G. Here, as an example we calculate the products and explicit ring structure of LSU_2/T using these ideas.

Note that these results grew out a chapter of the author's thesis [12].

2. The root system, Weyl group and Cartan matrix of the loop group LG.

We know from compact simply-connected semi-simple Lie theory that the complexified Lie algebra $\mathbf{g}_{\mathbb{C}}$ of the compact Lie group G has a decomposition under the adjoint action of the maximal torus T of G. Then, from [6], we have the following theorem.

Theorem 2.1. *There is a decomposition*

$$\mathbf{g}_{\mathbb{C}} = \mathbf{t}_{\mathbb{C}} \bigoplus_{\alpha} \mathbf{g}_{\alpha},$$

where $\mathbf{g_0} = \mathbf{t}_{\mathbb{C}}$ *is the complexified Lie algebra of* T*, and*

$$\mathbf{g}_{\alpha} = \{\xi \in \mathbf{g}_{\mathbb{C}} : \mathbf{t} \cdot \xi = \alpha(\mathbf{t})\xi \, \forall \mathbf{t} \in \mathbf{T}\}.$$

The homomorphisms $\alpha : T \to \mathbb{T}$ for which $\mathbf{g}_{\alpha} \neq \mathbf{0}$ are called the *roots* of G. They form a finite subset of the lattice $\breve{T} = \mathrm{Hom}(T, \mathbb{T})$. By analogy, the complexified Lie algebra $L\mathbf{g}_{\mathbb{C}}$ of the loop group LG has a decomposition

$$L\mathbf{g}_{\mathbb{C}} = \bigoplus_{\mathbf{k} \in \mathbb{Z}} \mathbf{g}_{\mathbb{C}} \cdot \mathbf{z}^{\mathbf{k}}$$

where $\mathbf{g}_{\mathbb{C}}$ is the complexified Lie algebra of G. This is the decomposition into eigenspaces of the rotation action of the circle group $\mathbb{T}$ on the loops. The rotation action commutes with the adjoint action of the constant loops G, and from [13], we have the following theorem.

Theorem 2.2. *There is a decomposition of* $L\mathbf{g}_{\mathbb{C}}$ *under the action of the maximal torus* T *of* G*,*

$$Lg_{\mathbb{C}} = \bigoplus_{\mathbf{k}\in\mathbb{Z}} \mathbf{g_0}\cdot\mathbf{z^k} \oplus \bigoplus_{(\mathbf{k},\alpha)} \mathbf{g}_\alpha\cdot\mathbf{z^k}.$$

The pieces in this decomposition are indexed by homomorphisms

$$(k,\alpha): \mathbb{T}\times T \to \mathbb{T}.$$

The homomorphisms $(k,\alpha)\in\mathbb{Z}\times\check{T}$ which occur in the decomposition are called the *roots* of LG.

defination 2.3. *The set of roots is called the* root system *of LG and denoted by $\widehat{\Delta}$.*

Let δ be $(0,1)$. Then

$$\widehat{\Delta} = \bigcup_{k\in\mathbb{Z}} (\Delta\cup\{0\}+k\delta) = \Delta\cup\{0\}+\mathbb{Z}\delta,$$

where Δ is the root system of G. The root system $\widehat{\Delta}$ is the union of real roots and imaginary roots:

$$\widehat{\Delta} = \widehat{\Delta}_{\mathrm{re}} \cup \widehat{\Delta}_{\mathrm{im}},$$

where

$$\begin{aligned}\widehat{\Delta}_{\mathrm{re}} &= \{(\alpha,n):\alpha\in\Delta,\, n\in\mathbb{Z}\}\\ \widehat{\Delta}_{\mathrm{im}} &= \{(0,r): r\in\mathbb{Z}\}.\end{aligned}$$

definition 2.4. *Let the rank of G be l. Then, the set of simple roots of LG is*

$$\{(\alpha_i,0):\alpha_i\in\Sigma\, for\ 1\le i\le l\}\cup\{(-\alpha_{l+1},1)\},$$

where α_{l+1} is the highest weight of the adjoint representation of G.

The root system $\widehat{\Delta}$ can be divided into three parts as the positive and the negative and 0:

$$\widehat{\Delta} = \widehat{\Delta}^+ \cup \{0\} \cup \widehat{\Delta}^-$$

where

$$\begin{aligned} \widehat{\Delta}^+ &= \widehat{\Delta}^+_{\text{re}} \cup \widehat{\Delta}^+_{\text{im}}, \\ \widehat{\Delta}^- &= \widehat{\Delta}^-_{\text{re}} \cup \widehat{\Delta}^-_{\text{im}}, \end{aligned}$$

where

$$\begin{aligned} \widehat{\Delta}^+_{\text{re}} &= \{(\alpha, n) \in \widehat{\Delta}_{\text{re}} : n > 0\} \cup \{(\alpha, 0) : \alpha \in \Delta^+\}, \\ \widehat{\Delta}^+_{\text{im}} &= \{n\delta : n > 0\} \end{aligned}$$

and

$$\begin{aligned} \widehat{\Delta}^-_{\text{re}} &= -\widehat{\Delta}^+_{\text{re}}, \\ \widehat{\Delta}^-_{\text{im}} &= -\widehat{\Delta}^+_{\text{im}}. \end{aligned}$$

Now, we will give some examples. First, we will discuss the case of SU_2. The root system $\widehat{\Delta}$ of the loop group $LSU(2)$ has two basis elements $\mathbf{a_0} = (-\alpha, \mathbf{1})$ and $\mathbf{a_1} = (\alpha, \mathbf{0})$ where α is the simple root of SU_2. All roots of LSU_2 can be written as a sum of the simple roots $\mathbf{a_0}$ and $\mathbf{a_1}$.

Proposition 2.5. *The set of roots of LSU_2 is given by $\widehat{\Delta} = \widehat{\Delta}_{\text{re}} \cup \widehat{\Delta}_{\text{im}}$ where*

$$\begin{aligned} \widehat{\Delta}_{\text{re}} &= \{k\mathbf{a}_0 + l\mathbf{a}_1 : |k - l| = 1, k \in \mathbb{Z}\}, \\ \widehat{\Delta}_{\text{im}} &= \{k\mathbf{a_0} + \mathbf{ka_1} : \mathbf{k} \in \mathbb{Z}\}. \end{aligned}$$

corollary 2.6. *The set of positive roots of LSU_2 is given by*
$\widehat{\Delta}^+ = \widehat{\Delta}^+_{\text{re}} \cup \widehat{\Delta}^+_{\text{im}}$ *where*

$$\begin{aligned}\widehat{\Delta}^+_{\rm re} &= \{k\mathbf{a}_0 + l\mathbf{a}_1 : |k-l| = 1, k \in \mathbb{Z}^+\} \quad = \{(\alpha, r), (-\alpha, s) : r \geq 0, s > 0\}, \\ \widehat{\Delta}^+_{\rm im} &= \{k\mathbf{a_0} + \mathbf{k}\mathbf{a_1} : \mathbf{k} \in \mathbb{Z}^+\} \quad .\end{aligned}$$

In the case of LSU_n, for $n \geq 3$, the root system $\widehat{\Delta}$ of the loop group LSU_n has basis elements $\mathbf{a_0} = (-\alpha_\mathbf{0}, \mathbf{1})$ and $\mathbf{a_i} = (\alpha_\mathbf{i}, \mathbf{0})$, $\mathbf{1} \leq \mathbf{i} \leq \mathbf{n-1}$ where α_i is the simple root of SU_n and $\alpha_0 = \sum_{i=1}^{n-1} \alpha_i$. All roots of LSU_n can be written as a sum of the simple roots $\mathbf{a_i}$.

Theorem 2.7. *(see [8])*

The set of roots of LSU_n, for $n \geq 3$, is

$$\widehat{\Delta} = \{k\sum_{r=0}^{i-1} \mathbf{a_r} + \mathbf{l}\sum_{\mathbf{r=i}}^{\mathbf{j-1}} \mathbf{a_r} + \mathbf{k}\sum_{\mathbf{r=j}}^{\mathbf{n-1}} \mathbf{a_r} : |\mathbf{k}-\mathbf{l}| = \mathbf{1}, \mathbf{k} \in \mathbb{Z} \quad \textit{and} \quad \mathbf{0} \leq \mathbf{i} \leq \mathbf{j} \leq \mathbf{n}\}.$$

Corollary 2.8. *The set of positive roots of LSU_n, for $n \geq 3$, is*

$$\widehat{\Delta}^+ = \{k\sum_{r=0}^{i-1} \mathbf{a_r} + \mathbf{l}\sum_{\mathbf{r=i}}^{\mathbf{j-1}} \mathbf{a_r} + \mathbf{k}\sum_{\mathbf{r=j}}^{\mathbf{n-1}} \mathbf{a_r} : |\mathbf{k}-\mathbf{l}| = \mathbf{1}, \mathbf{k} \in \mathbb{Z}^+ \quad \textit{and} \quad \mathbf{0} \leq \mathbf{i} \leq \mathbf{j} \leq \mathbf{n}\}.$$

Now, we will discuss the Weyl group of the loop group LG. In order to define this group, we need a larger group structure. We define the semi-direct product $\mathbb{T} \ltimes LG$ of $\mathbb{T}$ and LG in which $\mathbb{T}$ acts on LG by the rotation. From [13], we have the following two theorems.

Theorem 2.9. *$\mathbb{T} \times T$ is a maximal abelian subgroup of $\mathbb{T} \ltimes LG$.*

Theorem 2.10. *The complexified Lie algebra of $\mathbb{T} \ltimes LG$ has a decomposition*

$$(\mathbb{C} \oplus \mathbf{t}_\mathbb{C}) \oplus \left(\bigoplus_{k \neq 0} \mathbf{t}_\mathbb{C} \cdot \mathbf{z^k} \oplus \bigoplus_{(\mathbf{k}, \alpha)} \mathbf{g}_\alpha \cdot \mathbf{z^k}, \right)$$

according to the characters of $\mathbb{T} \times T$.

We know that the roots of G are permuted by the Weyl group W. This is the group of automorphisms of the maximal torus T which arise from conjugation in G, i.e. $W = N(T)/T$, where

$$N(T) = \{n \in G : nTn^{-1} = T\}$$

is the normalizer of T in G. In exactly same way, the infinite set of roots of LG is permuted by the Weyl group $\widetilde{W} = N(\mathbb{T} \times T)/(\mathbb{T} \times T)$, where $N(\mathbb{T} \times T)$ is the normalizer in $\mathbb{T} \ltimes LG$. The Weyl group $\widetilde{W}$ which was defined above is called the *affine Weyl group.*

Proposition 2.11. *The affine Weyl group $\widetilde{W}$ is the semidirect product of the coweight lattice $T^{\vee} = Hom(\mathbb{T}, T)$ by the Weyl group W of G.*

We know that the Weyl group W of G acts on the Lie algebra of the maximal torus T, it is a finite group of isometries of the Lie algebra $\mathbf{t}$ of the maximal torus T. It preserves the coweight lattice $T^{\vee}$. For each simple root α, the Weyl group W contains an element r_α of order two represented by $\exp\left(\frac{\pi}{2}(e_\alpha + e_{-\alpha})\right)$ in $N(T)$. Since the roots α can be considered as the linear functionals on the Lie algebra $\mathbf{t}$ of the maximal torus T, the action of r_α on $\mathbf{t}$ is given by

$$r_\alpha(\xi) = \xi - \alpha(\xi)h_\alpha \text{ for } \xi \in \mathbf{t},$$

where h_α is the coroot in $\mathbf{t}$ corresponding to simple root α. Also, we can give the action of r_α on the roots by

$$r_\alpha(\beta) = \beta - \alpha(h_\beta)\alpha \text{ for } \alpha, \beta \in \mathbf{t}^*,$$

where $\mathbf{t}^*$ is the dual vector space of $\mathbf{t}$. The element r_α is the reflection in the hyperplane H_α of $\mathbf{t}$ whose equation is $\alpha(\xi) = 0$. These reflections r_α generate the Weyl group W. For the special unitary matrix group SU_2, we have only one simple root α with corresponding reflection r_α which generates the Weyl group of SU_2 and $W \cong \mathbb{Z}/2$. More generally, we have from [7] this theorem:

Theorem 2.12. *The Weyl group of SU_n is the symmetric group S_n.*

Now, we want to describe the Weyl group structure of LG. By analogy with $\mathbb{R}$ for real form, the roots of the loop group LG can be considered as linear forms on the Lie algebra $\mathbb{R}\times\mathbf{t}$ of the maximal abelian group $\mathbb{T}\times T$. The Weyl group $\widetilde{W}$ acts linearly on $\mathbb{R}\times\mathbf{t}$, the action of W is an obvious reflection in the affine hyperplane $1\times\mathbf{t}$ and the action of $\lambda\in T^{\vee}$ is given by

$$\lambda\cdot(x,\xi)=(x,\xi+x\lambda).$$

Thus, the Weyl group $\widetilde{W}$ preserves the hyperplane $1\times\mathbf{h}$, and $\lambda\in\check{T}$ acts on it by translation by the vector $\lambda\in T^{\vee}\subset\mathbf{t}$. If $\alpha\neq 0$, the affine hyperplane $H_{\alpha,k}$ can be defined as follows. For each root (α,k),

$$H_{\alpha,k}=\{\xi\in\mathbf{t}:\alpha(\xi)=-\mathbf{k}\}.$$

We know that the Weyl group W of G is generated by the reflections r_α in the hyperplanes H_α for the simple roots α. A corresponding statement holds for the affine Weyl group $\widetilde{W}$.

Proposition 2.13 *Let G be a simply-connected semi-simple compact Lie group. Then the Weyl group $\widetilde{W}$ of the loop group LG is generated by the reflections in the hyperplanes $H_{\alpha,k}$. The affine Weyl group $\widetilde{W}$ acts on the root system $\widehat{\Delta}$ by*

$$r_{(\alpha,k)}(\gamma,m)=(r_\alpha(\gamma),m-\alpha(h_\gamma)k)\,\text{for}\ (\alpha,k),(\gamma,m)\in\widehat{\Delta}.$$

Proposition 2.14 *The Weyl group $\widetilde{W}$ of LSU_2 is*

$$\widetilde{W}=\{(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^k,\,(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^k r_{\mathbf{a_0}},\,(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^k,\,(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^k r_{\mathbf{a_1}}:k\geq 0,\,r_{\mathbf{a_0}}^2=r_{\mathbf{a_1}}^2=Id\}.$$

Proposition 2.15 *The Weyl group of LSU_n is the semi-direct product $S_n\ltimes\mathbb{Z}^{n-1}$ where S_n acts by permutation action on coordinates of $\mathbb{Z}^{n-1}$.*

Actually the symmetric group S_n acts on $\mathbb{Z}^n$ by the permutation action. $\mathbb{Z}^{n-1}$ is the fixed subgroup which corresponds to the eigen-value action. From [5], we have

Theorem 2.16 *The affine Weyl group $\widetilde{W}$ of LG is a Coxeter group.*

We will give some properties of the affine Weyl group $\widetilde{W}$.

Definition 2.17 *The* length *of an element $w \in \widetilde{W}$ is the least number of factors in the decomposition relative to the set of the reflections $\{r_{\mathbf{a_i}}\}$, is denoted by $\ell(w)$.*

Definition 2.18 *Let $w_1, w_2 \in \widetilde{W}, \gamma \in \Delta^+_{\mathrm{re}}$. Then $w_1 \xrightarrow{\gamma} w_2$ indicates the fact that*

$$\begin{aligned} r_\gamma w_1 &= w_2, \\ \ell(w_2) &= \ell(w_1) + 1. \end{aligned}$$

We put $w \leqslant w'$ if there is a chain

$$w = w_1 \to w_2 \to \cdots \to w_k = w'.$$

The relation $\leqslant$ is called the Bruhat order *on the affine Weyl group $\widetilde{W}$.*

Proposition 2.19 *Let $w \in \widetilde{W}$ and let $w = r_{\mathbf{a_1}} r_{\mathbf{a_2}} \cdots r_{\mathbf{a_l}}$ be the reduced decomposition of w. If $1 \leq i_1 < \ldots < i_k \leq l$ and $w' = r_{\mathbf{a_{i_1}}} r_{\mathbf{a_{i_2}}} \cdots r_{\mathbf{a_{i_k}}}$, then $w' \leqslant w$. If $w' \leqslant w$, then w' can be represented as above for some indexing set $\{i_\xi\}$. If $w' \to w$, then there is a unique index i, $1 \leq i \leq l$ such that*

$$w' = r_{\mathbf{a_1}} \cdots r_{\mathbf{a_{i-1}}} r_{\mathbf{a_{i+1}}}.$$

The last proposition gives an alternative definition of the Bruhat ordering on $\widetilde{W}$. Now we will define the subset $\widehat{W}$ of the affine Weyl group $\widetilde{W}$ which will be used in the text later. We know that the Weyl group $\widetilde{W}$ of the loop group LG is a split extension $T^\vee \to \widetilde{W} \to W$, where W is the Weyl group of the compact group Lie group G. Since the Weyl group W is a sub-Coxeter system of the affine Weyl group $\widetilde{W}$, we can define the set of cosets $\widetilde{W}/W$.

Lemma 2.20 *The subgroup of $\widetilde{W}$ fixing 0 is the Weyl group W.*

Corollary 2.21. *Let $w, w' \in \widetilde{W}$. Then, $w(0) = w'(0)$ if and only if $wW = w'W$ in $\widetilde{W}/W$.*

By the last corollary, the map $\widetilde{W}/W \to T^\vee$ given by $wW \to w(0)$ is well-defined and has inverse map given by $\chi_i \to r_{\alpha_i}W$, so the coset set $\widetilde{W}/W$ is identified to $T^\vee$ as set. We have from [1],

Theorem 2.22. *Each coset in $\widetilde{W}/W$ has a unique element of the minimal length.*

We will write $\overline{\ell(w)}$ for the minimal length element occuring in the coset wW, for $w \in \widetilde{W}$. We see that each coset $wW, w \in \widetilde{W}$ has two distinguished representatives which are not in the general the same. Let the subset $\widehat{W}$ of the affine Weyl group $\widetilde{W}$ be the set of the minimal representative elements $\overline{\ell(w)}$ in the coset wW for each $w \in \widetilde{W}$. The subset $\widehat{W}$ has the Bruhat order since it identitifies the set of the minimal representative elements $\overline{\ell(w)}$. As a example, we calculate the subset $\widehat{W}$ of the Weyl group of LSU_2. Our aim is to find the minimal representative elements $\overline{\ell(w)}$ in the right coset wW for each the element $w \in \widetilde{W}$, where

$$\widetilde{W} = \{(r_{\mathbf{a_0}} r_{\mathbf{a_1}})^k,\, (r_{\mathbf{a_0}} r_{\mathbf{a_1}})^l r_{\mathbf{a_0}},\, (r_{\mathbf{a_1}} r_{\mathbf{a_0}})^m,\, (r_{\mathbf{a_1}} r_{\mathbf{a_0}})^n r_{\mathbf{a_1}} : k, l, m, n \geq 0,\, r_{\mathbf{a_0}}^2 = r_{\mathbf{a_1}}^2 = \mathrm{id}\},$$

and $W = \langle r_{\mathbf{a_1}};\, r_{\mathbf{a_1}}^2 = \mathrm{id}\rangle$. We have the minimal representative elements $\overline{\ell(w)}$ for each coset wW, $w \in \widetilde{W}$ as follows

$$\begin{aligned}
\overline{l((r_{\mathbf{a_0}} r_{\mathbf{a_1}})^k)} &= (r_{\mathbf{a_0}} r_{\mathbf{a_1}})^k \quad \text{for } k \geq 0 \\
\overline{l((r_{\mathbf{a_0}} r_{\mathbf{a_1}})^l r_{\mathbf{a_0}})} &= (r_{\mathbf{a_0}} r_{\mathbf{a_1}})^l r_{\mathbf{a_0}} \quad \text{for } l \geq 0 \\
\overline{l((r_{\mathbf{a_1}} r_{\mathbf{a_0}})^n r_{\mathbf{a_1}})} &= (r_{\mathbf{a_0}} r_{\mathbf{a_1}})^n \quad \text{for } n \geq 0
\end{aligned}$$

and

$$\overline{l((r_{\mathbf{a_1}} r_{\mathbf{a_0}})^m)} = \begin{cases} \mathrm{Id} & \text{for m} = 0 \\ (r_{\mathbf{a_0}} r_{\mathbf{a_1}})^{m-1} r_{\mathbf{a_0}} & \text{for } m > 0 \end{cases}$$

By the transformations $m-1, l$ and $k \to n$, we have the subset

$$\widehat{W} = \{\overline{\ell(w)} : w \in \widetilde{W}\} = \{(r_{\mathbf{a_0}} r_{\mathbf{a_1}})^n,\ (r_{\mathbf{a_0}} r_{\mathbf{a_1}})^n r_{\mathbf{a_0}} : n \geq 0\}.$$

Now we will describe the Lie algebra $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ and its universal central extension in terms of generators and relations. For a finite dimensional semi-simple Lie algebra $\mathbf{g}_{\mathbb{C}}$, we can choose a non-zero element e_α in $\mathbf{g}_\alpha$ for each root α. From [6], we have

Theorem 2.23. *$\mathbf{g}_{\mathbb{C}}$ is a Kač-Moody Lie algebra generated by $e_i = e_{\alpha_i}$ and $f_i = e_{-\alpha_i}$ for $i = 1, \ldots, l$ where the α_i are the simple roots and l is the rank of $\mathbf{g}_{\mathbb{C}}$ only if G is semi-simple.*

Let us choose generators e_j and f_j of $L\mathbf{g}_{\mathbb{C}}$ corresponding to simple affine roots. Since $\mathbf{g}_{\mathbb{C}} \subset \mathbf{Lg}_{\mathbb{C}}$, we can take

$$e_j = \begin{cases} z e_{-\alpha_0} & \text{for } j = 0, \\ e_i & \text{for } 1 \leq j \leq l \end{cases}$$

and

$$f_j = \begin{cases} z^{-1} e_{\alpha_0} & \text{for } j = 0, \\ f_i & \text{for } 1 \leq j \leq l \end{cases}$$

where α_0 is the highest root of the adjoint representation. From [13],

Theorem 2.24. *Let $\mathbf{g}_{\mathbb{C}}$ be a semi-simple Lie algebra. Then, $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ is generated by the elements e_j and f_j corresponding to simple affine roots.*

The Cartan matrix $A_{(l+1)\times(l+1)}$ of $L\mathbf{g}_{\mathbb{C}}$ has the Cartan integers $a_{ij} = \mathbf{a_j}(\mathbf{h_{a_i}})$ as the entries where $\mathbf{a_0} = -\alpha_{\mathbf{0}}$, and $\mathbf{a_j} = \alpha_{\mathbf{j}}$ if $1 \leq j \leq l$. As an example,

Theorem 2.25. *Let $G = SU_2$. The Cartan matrix $A_{2\times 2}$ of $L\mathbf{g}_{\mathbb{C}}$ is the symmetric matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.*

Although the relations of the Kač-Moody algebra hold in $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$, they do not define it. By a theorem of Gabber and Kač in [2], the relations define the universal central extension $\widehat{L}_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ of $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ by $\mathbb{C}$ which is described by the cocycle ω_K given by

$$\omega_K(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \sigma(\xi(\theta),\eta'(\theta))d\theta.$$

As a vector space $\widehat{L}_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ is $L_{\text{pol}}\mathbf{g}_{\mathbb{C}} \oplus \mathbb{C}$ and the bracket is given by

$$[(\xi,\lambda),(\eta,\mu)] = ([\xi,\eta],\omega_K(\xi,\eta)).$$

Theorem 2.26. *$\widehat{L}\mathbf{g}_{\mathbb{C}}$ is an affine Kač-Moody algebra.*

3.1. Some homotopy equivalences for the loop group LG and its homogeneous spaces.

From [3], we have

Theorem 3.1. *The compact group G is a deformation retract of $G_{\mathbb{C}}$ and so, the loop space LG is homotopic to the complexified loop space $LG_{\mathbb{C}}$.*

Now, we want to give a major result from [13]

Theorem 3.2. *The inclusion*

$$\iota : L_{\text{pol}}G_{\mathbb{C}} \to LG_{\mathbb{C}}$$

is a homotopy equivalence.

Now we will give some useful notations. The parabolic subgroup P of $L_{\text{pol}}G_{\mathbb{C}}$ is the set of maps $\mathbb{C} \to G_{\mathbb{C}}$ which have non-negative Laurent series expansions. Then $P = G_{\mathbb{C}}[z]$. The minimal parabolic subgroup B is the Iwahori subgroup

$$\{f \in P : f(0) \in \overline{B}\},$$

where $\overline{B}$ is the finite-dimensional Borel subgroup of G. Note also that the minimal parabolic subgroup B corresponds to the positive roots, the parabolic subgroup P to the roots (α,n) with $n \geq 0$. From [3],

Theorem 3.3. *The evaluation at zero map $e_0 : P \to G_{\mathbb{C}}$ is a homotopy equivalence with the homotopy inverse the inclusion of $G_{\mathbb{C}}$ as the constant loops.*

The following fact follows from the local rigidity of the trivial bundle on the projective line. From [4], we have

Theorem 3.4. *The projection*

$$L_{\text{pol}}G_{\mathbb{C}} \to L_{\text{pol}}G_{\mathbb{C}}/P$$

is a principal bundle with fiber P.

Now, as a consequence of Theorem 3.2, Proposition 3.4 and Theorem 3.3, we have

Theorem 3.5. *$\Omega G_{\mathbb{C}}$ is homotopy equivalent to $L_{\text{pol}}G_{\mathbb{C}}/P$.*

Theorem 3.6. *(see [11]) The homogeneous space*

$$L_{\text{pol}}G_{\mathbb{C}}/P = \coprod_{w \in \widetilde{W}/W} BwP/P.$$

Corollary 3.7. *The homogeneous space*

$$L_{\text{pol}}G_{\mathbb{C}}/B = \coprod_{w \in \widetilde{W}} BwB/B.$$

By a theorem of [13], we have an isomorphism

Theorem 3.8.

$$H^*(LG/T; \mathbb{C}) \cong H^*(L\mathbf{g}_{\mathbb{C}}, \mathbf{t}_{\mathbb{C}}; \mathbb{C}) \cong \mathbf{H}^*(\widehat{\mathbf{L}}\mathbf{g}_{\mathbb{C}}, \widehat{\mathbf{t}}_{\mathbb{C}}; \mathbb{C}) \cong \mathbf{H}^*(\widehat{\mathbf{L}}_{\text{pol}}\mathbf{G}_{\mathbb{C}}/\widehat{\mathbf{B}}; \mathbb{C}).$$

By Theorem 3.8, the $\mathbb{Z}$-cohomology ring of LG/T generated by the strata can be calculated using a corollary of [9]. In the next section, we will work at an example.

4. Cohomology rings of the homogeneous spaces ΩSU_2 and LSU_2/T.

In order to determine the integral cohomology ring of LSU_2/T, we need some calculations in the integral cohomology of LSU_2/T.

Theorem 4.1. *For $n \geq 0$, the action of affine Weyl group of LSU_2 on the real root system is given by*

$$\begin{aligned}
(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n(-\alpha,s) &= (-\alpha,s+2n); && (4.1)\\
(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n(\alpha,r) &= (\alpha,r-2n), && (4.2)\\
(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n r_{\mathbf{a_0}}(-\alpha,s) &= (\alpha,s-2n-2); && (4.3)\\
(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n r_{\mathbf{a_0}}(\alpha,r) &= (-\alpha,r+2n+2), && (4.4)\\
(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n(-\alpha,s) &= (-\alpha,s-2n); && (4.5)\\
(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n(\alpha,r) &= (\alpha,r+2n), && (4.6)\\
(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n r_{\mathbf{a_1}}(-\alpha,s) &= (\alpha,s+2n); && (4.7)\\
(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n r_{\mathbf{a_1}}(\alpha,r) &= (-\alpha,r-2n). && (4.8)
\end{aligned}$$

Proof. First, by induction on n, we shall show that

$$\begin{aligned}
(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^n(-\alpha,s) &= (-\alpha,s+2n)\\
(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^n(\alpha,r) &= (\alpha,r-2n),
\end{aligned}$$

for $(-\alpha,s)$, $(\alpha,r) \in \widehat{\Delta}_{\text{re}}$. The case $n=0$ is trivially true.

Now, we assume that the equations Eq.(4.1) and Eq.(4.2) hold for $n=l$. Then,

$$\begin{aligned}
(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^{l+1}(-\alpha,s) &= (r_{\mathbf{a}_0}r_{\mathbf{a}_1})(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^l(-\alpha,s)\\
&= (r_{\mathbf{a}_0}r_{\mathbf{a}_1})(-\alpha,s+2l)\\
&= r_{\mathbf{a}_0}(\alpha,s+2l)\\
&= (-\alpha,s+2(l+1)),
\end{aligned}$$

and

$$\begin{aligned}(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^{l+1}(\alpha,r) &= (r_{\mathbf{a}_0}r_{\mathbf{a}_1})(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^{l}(\alpha,r)\\ &= (r_{\mathbf{a}_0}r_{\mathbf{a}_1})(\alpha,r-2l)\\ &= r_{\mathbf{a}_0}(-\alpha,r-2l)\\ &= (\alpha,r-2(l+1)).\end{aligned}$$

This means that Equations Eq(4.1) and Eq(4.2) hold for any $n \geq 0$.

Since $(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n r_{\mathbf{a}_1} = r_{\mathbf{a}_1}(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^n$, we can find easily the action of the reflection $(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}$ on the real root system.

Then, we have Equation Eq.(4.7) and Eq.(4.8),

$$(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}(-\alpha,s) = r_{\mathbf{a}_1}(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^n(-\alpha,s) = r_{\mathbf{a}_1}(-\alpha,s+2n) = (\alpha,s+2n),$$

and

$$(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}(\alpha,r) = r_{\mathbf{a}_1}(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^n(\alpha,r) = r_{\mathbf{a}_1}(\alpha,r-2n) = (-\alpha,r-2n).$$

Since $(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n$ is inverse of $(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^n$, the action of $(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n$ on the real root system is given by

$$\begin{aligned}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n(\alpha,r) &= (\alpha,r+2n)\\ (r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n(-\alpha,s) &= (-\alpha,s-2n).\end{aligned}$$

Also, since $(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^n r_{\mathbf{a}_0} = r_{\mathbf{a}_0}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n$, the action of $(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^n r_{\mathbf{a}_0}$ on the real root system is given by

$$(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^n r_{\mathbf{a}_0}(\alpha,r) = r_{\mathbf{a}_0}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n(\alpha,r) = r_{\mathbf{a}_0}(\alpha,r+2n) = (-\alpha,r+2n+2),$$

and

$$(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^n r_{\mathbf{a}_0}(-\alpha,s) = r_{\mathbf{a}_0}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^n(-\alpha,s) = r_{\mathbf{a}_0}(-\alpha,s-2n) = (\alpha,s-2n-2).$$

□

Corollary 4.2. *Let* (α,u) *and* $(-\alpha,v), u\geq 0, v>0$, *be real positive roots of* LSU_2. *For* $n\geq 0$,

$$\begin{aligned}
r_{(\alpha,u)}(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n(-\alpha,s) &= (\alpha,s+2n+2u); && (4.9)\\
r_{(\alpha,u)}(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n(\alpha,r) &= (-\alpha,r-2n-2u), && (4.10)\\
r_{(\alpha,u)}(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n r_{\mathbf{a_0}}(-\alpha,s) &= (-\alpha,s-2n-2u-2); && (4.11)\\
r_{(\alpha,u)}(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n r_{\mathbf{a_0}}(\alpha,r) &= (\alpha,r+2n+2u+2), && (4.12)\\
r_{(\alpha,u)}(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n(-\alpha,s) &= (\alpha,s-2n+2u); && (4.13)\\
r_{(\alpha,u)}(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n(\alpha,r) &= (-\alpha,r+2n-2u), && (4.14)\\
r_{(\alpha,u)}(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n r_{\mathbf{a_1}}(-\alpha,s) &= (-\alpha,s+2n-2u); && (4.15)\\
r_{(\alpha,u)}(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n r_{\mathbf{a_1}}(\alpha,r) &= (\alpha,r-2n+2u), && (4.16)
\end{aligned}$$

$$\begin{aligned}
r_{(-\alpha,v)}(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n(-\alpha,s) &= (\alpha,s+2n-2v); && (4.17)\\
r_{(-\alpha,v)}(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n(\alpha,r) &= (-\alpha,r-2n+2v), && (4.18)\\
r_{(-\alpha,v)}(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n r_{\mathbf{a_0}}(-\alpha,s) &= (-\alpha,s-2n+2v-2); && (4.19)\\
r_{(-\alpha,v)}(r_{\mathbf{a_0}}r_{\mathbf{a_1}})^n r_{\mathbf{a_0}}(\alpha,r) &= (\alpha,r+2n-2v+2), && (4.20)\\
r_{(-\alpha,v)}(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n(-\alpha,s) &= (\alpha,s-2n-2v); && (4.21)\\
r_{(-\alpha,v)}(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n(\alpha,r) &= (-\alpha,r+2n+2v), && (4.22)\\
r_{(-\alpha,v)}(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n r_{\mathbf{a_1}}(-\alpha,s) &= (-\alpha,s+2n+2v); && (4.23)\\
r_{(-\alpha,v)}(r_{\mathbf{a_1}}r_{\mathbf{a_0}})^n r_{\mathbf{a_1}}(\alpha,r) &= (\alpha,r-2n-2v). && (4.24)
\end{aligned}$$

Theorem 4.3. *For* $k\geq 0$, *the following equations hold in* $H^*(LSU_2/T,\mathbb{Z})$.

$$\begin{aligned}
(\varepsilon^{r_{\mathbf{a}_0}})^{2k} &= (2k)!\,\varepsilon^{(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^k}, && (4.25)\\
(\varepsilon^{r_{\mathbf{a}_1}})^{2k} &= (2k)!\,\varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^k}, && (4.26)\\
(\varepsilon^{r_{\mathbf{a}_0}})^{2k+1} &= (2k+1)!\,\varepsilon^{(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^k r_{\mathbf{a}_0}}, && (4.27)\\
(\varepsilon^{r_{\mathbf{a}_1}})^{2k+1} &= (2k+1)!\,\varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^k r_{\mathbf{a}_1}} && (4.28)
\end{aligned}$$

Proof. By induction on k, we will show that these equations hold in $H^*(LSU_2/T, \mathbb{Z})$. For $k = 0$, these equations hold.

Now, we assume that these equations hold for $k = n$. Then, we have to show that they hold for $k = n + 1$. By assumption,

$$\begin{aligned}(\varepsilon^{r_{\mathbf{a}_0}})^{2n+2} &= (\varepsilon^{r_{\mathbf{a}_0}}) \cdot (\varepsilon^{r_{\mathbf{a}_0}})^{2n+1} \\ &= (2n+1)!\, \varepsilon^{r_{\mathbf{a_0}}} \cdot \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n r_{\mathbf{a_0}}}.\end{aligned}$$

We have

$$(\varepsilon^{r_{\mathbf{a}_0}})^{2n+2} = (2n+1)! \sum_{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n r_{\mathbf{a}_0} \xrightarrow{\gamma} w} \chi_0(h_\gamma) \varepsilon^w.$$

When we check the action of the reflections which have length $2n+2$, by the action of $r_{(\alpha,u)}(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n r_{\mathbf{a}_0}$ and $r_{(-\alpha,v)}(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n r_{\mathbf{a}_0}$ on the real root system, we see that the sum in the right side of the last cup product equation holds the only for the positive root

$$(-\alpha, 2n+2) = (2n+2)\mathbf{a_0} + (2n+1)\mathbf{a_1}.$$

Then,

$$(\varepsilon^{r_{\mathbf{a}_0}})^{2n+2} = (2n+2)!\, \varepsilon^{r_{(-\alpha,2n+2)}(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n r_{\mathbf{a_0}}}.$$

The composition of reflections $r_{(-\alpha,2n+2)}(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n r_{\mathbf{a_0}}$ can be represented by the Weyl group element $(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n+1}$, so

$$(\varepsilon^{r_{\mathbf{a}_0}})^{2n+2} = (2n+2)!\, \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n+1}}.$$

If we continue the induction for equation Eq.(4.27), by assumption,

$$\begin{aligned}(\varepsilon^{r_{\mathbf{a}_0}})^{2n+3} &= (\varepsilon^{r_{\mathbf{a}_0}}) \cdot (\varepsilon^{r_{\mathbf{a}_0}})^{2n+2} \\ &= (2n+2)!\, \varepsilon^{r_{\mathbf{a_0}}} \cdot \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n+1}}.\end{aligned}$$

We have

$$(\varepsilon^{r_{\mathbf{a}_0}})^{2n+3} = (2n+2)! \sum_{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n+1} \xrightarrow{\gamma} w} \chi_0(h_\gamma)\varepsilon^w.$$

When we check the action of the reflections which have length $2n+3$, by the action of $r_{(\alpha,u)}(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n+1}$ and $r_{(-\alpha,v)}(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n+1}$ on the real root system, we see that the sum in the right side of the last cup product equation holds only for the positive root

$$(-\alpha, 2n+3) = (2n+3)\mathbf{a_0} + (2n+2)\mathbf{a_1}.$$

Then,

$$(\varepsilon^{r_{\mathbf{a}_0}})^{2n+3} = (2n+3)!\, \varepsilon^{r_{(-\alpha,2n+3)}(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n+1}}.$$

The composition of reflections $r_{(-\alpha,2n+3)}(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n+1}$ can be represented by the Weyl group element $(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n+1} r_{\mathbf{a_0}}$, so

$$(\varepsilon^{r_{\mathbf{a}_0}})^{2n+3} = (2n+3)!\, \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n+1} r_{\mathbf{a_0}}}.$$

Thus, we have proved that the equations Eq.(4.25) and Eq.(4.27) hold in $H^*(LSU_2/T, \mathbb{Z})$.

Similarly, by assumption,

$$\begin{aligned} (\varepsilon^{r_{\mathbf{a}_1}})^{2n+2} &= (\varepsilon^{r_{\mathbf{a}_1}}) \cdot (\varepsilon^{r_{\mathbf{a}_1}})^{2n+1} \\ &= (2n+1)!\, \varepsilon^{r_{\mathbf{a_1}}} \cdot \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n r_{\mathbf{a_1}}}. \end{aligned}$$

We have

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+2} = (2n+1)! \sum_{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n r_{\mathbf{a}_1} \xrightarrow{\gamma} w} \chi_1(h_\gamma)\varepsilon^w.$$

When we check the action of the reflections which have length $2n+2$, by the action of $r_{(\alpha,u)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}$ and $r_{(-\alpha,v)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}$ on the real root system, we see that the sum in the right side of the last cup product equation holds the only for the positive root

$$(\alpha, 2n+1) = (2n+1)\mathbf{a_0} + (2n+2)\mathbf{a_1}.$$

Then,

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+2} = (2n+2)!\, \varepsilon^{r_{(\alpha,2n+1)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n r_{\mathbf{a_1}}}.$$

The composition of reflections $r_{(\alpha,2n+1)}(r_{\mathbf{a_1}} r_{\mathbf{a_0}})^n r_{\mathbf{a_1}}$ can be represented by the Weyl group element $(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1}$, so

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+2} = (2n+2)!\, \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1}}.$$

If we continue the induction for equation Eq.(4.28),

$$\begin{aligned} (\varepsilon^{r_{\mathbf{a}_1}})^{2n+3} &= (\varepsilon^{r_{\mathbf{a}_1}}) \cdot (\varepsilon^{r_{\mathbf{a}_1}})^{2n+2} \\ &= (2n+2)!\, \varepsilon^{r_{\mathbf{a_1}}} \cdot \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1}}. \end{aligned}$$

We have

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+3} = (2n+2)! \sum_{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1} \xrightarrow{\gamma} w} \chi_1(h_\gamma) \varepsilon^w.$$

When we check the action of the reflections which have length $2n+3$, by the action of $r_{(\alpha,u)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1}$ and $r_{(-\alpha,v)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1}$ on the real root system, we see that the sum in the right side of the last cup product equation holds the only for the positive root

$$(\alpha, 2n+2) = (2n+2)\mathbf{a_0} + (2n+3)\mathbf{a_1}.$$

Then,

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+3} = (2n+3)!\, \varepsilon^{r_{(\alpha,2n+2)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1}}.$$

The composition of reflections $r_{(\alpha,2n+2)}(r_{\mathbf{a}_1} r_{\mathbf{a_0}})^{n+1}$ can be represented by the Weyl group element $(r_{\mathbf{a}_1} r_{\mathbf{a_0}})^{n+1} r_{\mathbf{a_1}}$, so

$$(\varepsilon^{r_{\mathbf{a}_1}})^{2n+3} = (2n+3)!\, \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n+1} r_{\mathbf{a_1}}}.$$

So, the induction is completed and we have proved that all equations hold in $H^*(LSU_2/T, \mathbb{Z})$. □

We will make another calculation in the integral cohomology algebra of LSU_2/T.

Theorem 4.4. *For $n, m \geq 0$, the following equation holds in $H^*(LSU_2/T, \mathbb{Z})$.*

$$(n+m)(\varepsilon^{r_{\mathbf{a}_0}})^n \cdot (\varepsilon^{r_{\mathbf{a}_1}})^m = n(\varepsilon^{r_{\mathbf{a}_0}})^{n+m} + m(\varepsilon^{r_{\mathbf{a}_1}})^{n+m}.$$

Proof. By induction on m, we shall prove that the result holds in $H^*(LSU_2/T, \mathbb{Z})$. Since the integral cohomology ring of LSU_2/T is torsion-free, the integral cohomology ring can be embedded in the rational cohomology ring hence the calculations can be done in the rational cohomology. For $m = 0$, the equation obviously holds.

First, we will verify the equation for $m = 1$. For $m = 1$, the equation reduces to

$$(n+1)(\varepsilon^{r_{\mathbf{a}_0}})^n \cdot (\varepsilon^{r_{\mathbf{a}_1}}) = n(\varepsilon^{r_{\mathbf{a}_0}})^{n+1} + (\varepsilon^{r_{\mathbf{a}_1}})^{n+1}. \tag{4.29}$$

Now, we will use sub-induction with respect to n on the equation Eq.(4.29). The equation Eq.(4.29) obviously holds for $n = 0$.

Now, we assume that equation Eq.(4.29) holds for $n = k$. We verify that equation Eq.(4.29) holds for $n = k+1$. By the induction hypothesis, we have

$$\begin{aligned}
\varepsilon^{r_{\mathbf{a}_1}} \cdot (\varepsilon^{r_{\mathbf{a}_0}})^{k+1} &= (\varepsilon^{r_{\mathbf{a}_1}} \cdot (\varepsilon^{r_{\mathbf{a}_0}})^k) \cdot \varepsilon^{r_{\mathbf{a}_0}} \\
&= \left(\frac{k}{k+1}(\varepsilon^{r_{\mathbf{a}_0}})^{k+1} + \frac{1}{k+1}(\varepsilon^{r_{\mathbf{a}_1}})^{k+1}\right) \cdot \varepsilon^{r_{\mathbf{a}_0}} \\
&= \frac{k}{k+1}(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{1}{k+1}(\varepsilon^{r_{\mathbf{a}_1}})^{k+1} \cdot \varepsilon^{r_{\mathbf{a}_0}}.
\end{aligned} \tag{4.30}$$

Now, we calculate the cup product

$$(\varepsilon^{r_{\mathbf{a}_1}})^{k+1} \cdot \varepsilon^{r_{\mathbf{a}_0}}$$

in the above equation. We now treat the case k odd or even separately. If $k = 2l-1$, by equation Eq.(4.26),

$$\varepsilon^{r_{\mathbf{a}_0}} \cdot (\varepsilon^{r_{\mathbf{a}_1}})^{2l} = (2l)!\left(\varepsilon^{r_{\mathbf{a}_0}} \cdot (\varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l})\right). \tag{4.31}$$

By the cup product formula,

$$\varepsilon^{r_{\mathbf{a}_0}} \cdot \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l} = \sum_{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l \xrightarrow{\gamma} w} \chi_0(h_\gamma)\varepsilon^w .$$

When we check the action of reflections $r_{(\alpha,u)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{l1}$ and $r_{(-\alpha,v)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l$ by the action of the Weyl group elements $(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}$ and $(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^l r_{\mathbf{a}_0}$ which has length $2l+1$, we see that the reflections $r_{(-\alpha,1)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l$ and $r_{(\alpha,2l)}(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l$ can be represented by the Weyl group elements $(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^l r_{\mathbf{a}_0}$ and $(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}$ respectively. Using the positive root $(\alpha, 2l) = (2l)\,\mathbf{a}_0 + (2l+1)\,\mathbf{a}_1$ in the cup product formula,

$$\varepsilon^{r_{\mathbf{a}_0}} \cdot \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l} = \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^l r_{\mathbf{a}_0}} + (2l)\,\varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}} . \tag{4.32}$$

By equations Eq.(4.27) and Eq.(4.28),

$$\varepsilon^{r_{\mathbf{a}_0}} \cdot \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l} = \frac{1}{(2l+1)!}(\varepsilon^{r_{\mathbf{a}_0}})^{2l+1} + \frac{2l}{(2l+1)!}(\varepsilon^{r_{\mathbf{a}_1}})^{2l+1}. \tag{4.33}$$

When the last result is placed in the equation Eq.(4.31), we have

$$\begin{aligned}
\varepsilon^{r_{\mathbf{a}_0}} \cdot (\varepsilon^{r_{\mathbf{a}_1}})^{2l} &= (2l)!\left(\varepsilon^{r_{\mathbf{a}_0}} \cdot (\varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l})\right) \\
&= (2l)!\left(\frac{1}{(2l+1)!}(\varepsilon^{r_{\mathbf{a}_0}})^{2l+1} + \frac{2l}{(2l+1)!}(\varepsilon^{r_{\mathbf{a}_1}})^{2l+1}\right) \\
&= \frac{1}{2l+1}(\varepsilon^{r_{\mathbf{a}_0}})^{2l+1} + \frac{2l}{2l+1}(\varepsilon^{r_{\mathbf{a}_1}})^{2l+1}.
\end{aligned}$$

Using $k = 2l-1$, we have

$$\varepsilon^{r_{\mathbf{a}_0}} \cdot (\varepsilon^{r_{\mathbf{a}_1}})^{k+1} = \frac{1}{k+2}(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{k+1}{k+2}(\varepsilon^{r_{\mathbf{a}_1}})^{k+2}. \tag{4.34}$$

When the last result is placed in equation Eq.(4.30), we have

$$\begin{aligned}
\varepsilon^{r_{\mathbf{a}_1}} \cdot (\varepsilon^{r_{\mathbf{a}_0}})^{k+1} &= \frac{k}{k+1}\,(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{1}{k+1}\,(\varepsilon^{r_{\mathbf{a}_1}})^{k+1} \cdot \varepsilon^{r_{\mathbf{a}_0}} \\
&= \frac{k}{k+1}\,(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{1}{k+1}\,\left(\frac{1}{k+2}(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{k+1}{k+2}(\varepsilon^{r_{\mathbf{a}_1}})^{k+2}\right)
\end{aligned}$$

$$\begin{aligned} &= \left(\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\right)(\varepsilon^{r_{\mathbf{a}_0}})^{k+2}+\frac{1}{k+2}(\varepsilon^{r_{\mathbf{a}_1}})^{k+2} \\ &= \frac{k+1}{k+2}(\varepsilon^{r_{\mathbf{a}_0}})^{k+2}+\frac{1}{k+2}(\varepsilon^{r_{\mathbf{a}_1}})^{k+2}. \end{aligned}$$

If $k=2l$, by the equation Eq.(4.28),

$$(\varepsilon^{r_{\mathbf{a}_0}})\cdot(\varepsilon^{r_{\mathbf{a}_1}})^{2l+1}=(2l+1)!\left((\varepsilon^{r_{\mathbf{a}_0}})\cdot(\varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}})\right). \tag{4.35}$$

By the cup product formula,

$$(\varepsilon^{r_{\mathbf{a}_0}})\cdot(\varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}})=\sum_{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}\xrightarrow{\gamma}w}\chi_0(h_\gamma)\varepsilon^w.$$

When we check the action of reflections $r_{(\alpha,u)}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}$ and $r_{(-\alpha,v)}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}$ by the action of the Weyl group elements $(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^{l+1}$ and $(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^{l+1}$, which has length $2l+2$, we see that the reflections $r_{(-\alpha,1)}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}$ and $r_{(\alpha,2l+1)}(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}$ can be represented by the Weyl group elements $(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^{l+1}$ and $(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^{l+1}$ respectively. Using the positive root $(\alpha,2l+1)=(2l+1)\,\mathbf{a}_0+(2l+2)\,\mathbf{a}_1$, we have

$$(\varepsilon^{r_{\mathbf{a}_0}})\cdot(\varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}})=\varepsilon^{(r_{\mathbf{a}_0}r_{\mathbf{a}_1})^{l+1}}+(2l+1)\,\varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^{l+1}}. \tag{4.36}$$

By equations Eq.(4.25) and Eq.(4.26),

$$\varepsilon^{r_{\mathbf{a}_0}}\cdot\varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}}=\frac{1}{(2l+2)!}(\varepsilon^{r_{\mathbf{a}_0}})^{2l+2}+\frac{2l+1}{(2l+2)!}(\varepsilon^{r_{\mathbf{a}_1}})^{2l+2}. \tag{4.37}$$

When the last result is placed in the equation Eq.(4.35), we have

$$\begin{aligned} \varepsilon^{r_{\mathbf{a}_0}}\cdot(\varepsilon^{r_{\mathbf{a}_1}})^{2l+1} &= (2l+1)!\left(\varepsilon^{r_{\mathbf{a}_0}}\cdot(\varepsilon^{(r_{\mathbf{a}_1}r_{\mathbf{a}_0})^l r_{\mathbf{a}_1}})\right) \\ &= (2l+1)!\left(\frac{1}{(2t+2)!}(\varepsilon^{r_{\mathbf{a}_0}})^{2l+2}+\frac{2l+1}{(2l+2)!}(\varepsilon^{r_{\mathbf{a}_1}})^{2l+2}\right) \\ &= \frac{1}{2l+2}(\varepsilon^{r_{\mathbf{a}_0}})^{2l+2}+\frac{2l+1}{2l+2}(\varepsilon^{r_{\mathbf{a}_1}})^{2l+2}. \end{aligned}$$

Using $k=2l$, we have

$$\varepsilon^{r_{\mathbf{a}_0}} \cdot (\varepsilon^{r_{\mathbf{a}_1}})^{k+1} = \frac{1}{k+2}(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{k+1}{k+2}(\varepsilon^{r_{\mathbf{a}_1}})^{k+2}. \tag{4.38}$$

When the last result is placed in the equation Eq.(4.30), we have

$$\begin{aligned}
\varepsilon^{r_{\mathbf{a}_1}} \cdot (\varepsilon^{r_{\mathbf{a}_0}})^{k+1} &= \frac{k}{k+1}(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{1}{k+1}(\varepsilon^{r_{\mathbf{a}_1}})^{k+1} \cdot \varepsilon^{r_{\mathbf{a}_0}} \\
&= \frac{k}{k+1}(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{1}{k+1}\left(\frac{1}{k+2}(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{k+1}{k+2}(\varepsilon^{r_{\mathbf{a}_1}})^{k+2}\right) \\
&= \left(\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}\right)(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{1}{k+2}(\varepsilon^{r_{\mathbf{a}_1}})^{k+2} \\
&= \frac{k+1}{k+2}(\varepsilon^{r_{\mathbf{a}_0}})^{k+2} + \frac{1}{k+2}(\varepsilon^{r_{\mathbf{a}_1}})^{k+2}.
\end{aligned}$$

The induction on n is completed. Thus, we proved that the equation holds for $m = 1$.

We assume that equation holds for $m = s$. Then, we will verify that it holds for $m = s + 1$. By assumption,

$$\begin{aligned}
(\varepsilon^{r_{\mathbf{a}_0}})^n \cdot (\varepsilon^{r_{\mathbf{a}_1}})^{s+1} &= \left(\frac{n}{n+s}(\varepsilon^{r_{\mathbf{a}_0}})^{n+s} + \frac{s}{n+s}(\varepsilon^{r_{\mathbf{a}_1}})^{n+s}\right) \cdot \varepsilon^{r_{\mathbf{a}_1}} \\
&= \frac{n}{n+s}(\varepsilon^{r_{\mathbf{a}_0}})^{n+s} \cdot \varepsilon^{r_{\mathbf{a}_1}} + \frac{s}{n+s}(\varepsilon^{r_{\mathbf{a}_1}})^{n+s+1} \\
&= \frac{n}{n+s}\left(\frac{n+s}{n+s+1}(\varepsilon^{r_{\mathbf{a}_0}})^{n+s+1} + \frac{1}{n+s+1}(\varepsilon^{r_{\mathbf{a}_1}})^{n+s+1}\right) + \\
&\quad \frac{s}{n+s}(\varepsilon^{r_{\mathbf{a}_1}})^{n+s+1} \\
&= \frac{n}{n+s+1}(\varepsilon^{r_{\mathbf{a}_0}})^{n+s+1} + \left(\frac{n}{(n+s).(n+s+1)} + \frac{s}{n+s}\right)(\varepsilon^{r_{\mathbf{a}_1}})^{n+s+1} \\
&= \frac{n}{n+s+1}(\varepsilon^{r_{\mathbf{a}_0}})^{n+s+1} + \frac{s^2 + s(n+1) + n}{(n+s)(n+s+1)}(\varepsilon^{r_{\mathbf{a}_1}})^{n+s+1} \\
&= \frac{n}{n+s+1}(\varepsilon^{r_{\mathbf{a}_0}})^{n+s+1} + \frac{s+1}{(n+s+1)}(\varepsilon^{r_{\mathbf{a}_1}})^{n+s+1}.
\end{aligned}$$

Thus, the induction is completed. □

Let R be a commutative ring with unit and let $\Gamma_R(x_0, x_1)$ be the divided power algebra over R, where $\deg x_0 = \deg x_1 = 2$.

Theorem 4.5. *Then, $H^*(LSU_2/T,R)$ is graded isomorphic to $\Gamma_R(x_0,x_1)/I_R$ where the ideal I_R is given by*

$$I_R = \left(x_0^{[n]}x_1^{[m]} - \binom{n+m-1}{m} x_0^{[n+m]} - \binom{n+m-1}{n} x_1^{[n+m]} \,:\, m,n \geq 1 \right),$$

and which has the R-module basis $\{x_0^{[n]}, x_1^{[n]}\}$ in each degree $2n$ for $n \geq 1$.

Proof. Since the odd dimensional cohomology is trivial, by the universal coefficient theorem, it suffices to prove this for $R=\mathbb{Z}$. The Schubert classes $\{\varepsilon^w\}_{w\in\widetilde{W}_{LSU(2)}}$ form a basis of the integral cohomology $H^*(LSU_2/T,\mathbb{Z})$ such that $\varepsilon^w \in H^{2\ell(w)}(LSU_2/T,\mathbb{Z})$. Since the cohomology module basis is indexed by the affine Weyl group $\widetilde{W}$, the Poincaré series over $\mathbb{Z}$ of cohomology of LSU_2/T is

$$P(t,\mathbb{Z}) = 1 + \sum_{k=1}^{\infty} 2t^{2k}.$$

Now we will show that the integral cohomology algebra $H^*(LSU_2/T,\mathbb{Z})$ is isomorphic to the quotient of divided power algebra $\Gamma_{\mathbb{Z}}(x_0,x_1)/I_{\mathbb{Z}}$. Then, we define a $\mathbb{Z}$-algebra homomorphism ψ from the divided power algebra $\Gamma_{\mathbb{Z}}(x_0,x_1)$ to the integral cohomology of LSU_2/T as follows.

For $U = \sum_{i=0}^{n} u_i\, x_0^{[i]} x_1^{[n-i]}$ with $u_i \in \mathbb{Z}$, let

$$\psi(U) = u_n X(n) + u_0 Y(n) + \sum_{i=1}^{n-1} \left[\binom{n-1}{n-i} X(n) + \binom{n-1}{i} Y(n) \right] u_i,$$

where

$$X(n) = \begin{cases} \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^l} & \text{for } n=2l \\ \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^l r_{\mathbf{a}_0}} & \text{for } n=2l+1 \end{cases}$$

$$Y(n) = \begin{cases} \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^l} & \text{for } n=2l \\ \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^r_{\mathbf{a}_1}} & \text{for } n=2l+1. \end{cases}$$

We will show that ψ is a $\mathbb{Z}$-algebra homomorphism. Let

$$U = \sum_{i=0}^{n} u_i \, x_0^{[i]} x_1^{[n-i]} \quad V = \sum_{j=0}^{m} v_j \, x_0^{[j]} x_1^{[m-j]},$$

where $u_i, v_j \in \mathbb{Z}$. First, let us calculate

$$\begin{aligned}
\psi(U)\cdot\psi(V) &= \psi\left(\sum_{i=0}^{n} u_i \, x_0^{[i]} x_1^{[n-i]}\right)\cdot\psi\left(\sum_{j=0}^{m} v_j \, x_0^{[j]} x_1^{[m-j]}\right)\\
&= \left(u_0Y(n)+u_nX(n)+\sum_{i=1}^{n-1}u_i\left[\binom{n-1}{i-1}X(n)+\binom{n-1}{i}Y(n)\right]\right)\cdot\\
&\quad \left(v_0Y(m)+v_mX(m)+\sum_{j=1}^{m-1}v_j\left[\binom{m-1}{j-1}X(m)+\binom{m-1}{j}Y(m)\right]\right)\\
&= u_0v_0Y(n)Y(m)+u_0v_mY(n)X(m)+\sum_{j=1}^{m-1}u_0v_j\left[\binom{m-1}{j-1}Y(n)X(m)+\binom{m-1}{j}Y(n)Y(m)\right]\\
&+ u_nv_0X(n)Y(m)+u_nv_mX(n)X(m)+\sum_{j=1}^{m-1}u_nv_j\left[\binom{m-1}{j-1}X(n)X(m)+\binom{m-1}{j}X(n)Y(m)\right]\\
&+ \sum_{i=1}^{n-1}u_iv_0\left[\binom{n-1}{i-1}X(n)Y(m)+\binom{n-1}{i}Y(n)Y(m)\right]+\\
&\quad \sum_{i=1}^{n-1}u_iv_m\left[\binom{n-1}{i-1}X(n)X(m)+\binom{n-1}{i}Y(n)X(m)\right]\\
&+ \sum_{i=1}^{n-1}\sum_{j=1}^{m-1}u_iv_j\left[\binom{n-1}{i-1}\binom{m-1}{j-1}X(n)X(m)+\binom{n-1}{i-1}\binom{m-1}{j}X(n)Y(m)\right]\\
&+ \sum_{i=1}^{n-1}\sum_{j=1}^{m-1}u_iv_j\left[\binom{n-1}{i}\binom{m-1}{j-1}Y(n)X(m)+\binom{n-1}{i}\binom{m-1}{j}Y(n)Y(m)\right].
\end{aligned}$$

By equations Eq.(4.25), Eq.(4.26), Eq.(4.27), Eq.(4.28) and Eq.(4.4),

$$\begin{aligned}
Y(n)Y(m) &= \binom{n+m}{n}Y(n+m)\\
X(n)X(m) &= \binom{n+m}{n}X(n+m),
\end{aligned}$$

$$X(n)Y(m) = \binom{n+m-1}{m} X(n+m) + \binom{n+m-1}{n} Y(n+m)$$

and

$$Y(n)X(m) = \binom{n+m-1}{n} X(n+m) + \binom{n+m-1}{m} Y(n+m).$$

If we put the last results in the equation, we have

$$\psi(U)\cdot\psi(V) = X(n+m)\Bigg\{ u_0 v_m \binom{m+n-1}{n} + \sum_{j=1}^{m-1} u_0 v_j \binom{m-1}{j-1}\binom{m+n-1}{n} + u_n v_0$$
$$\binom{m+n-1}{m} + u_n v_m \binom{n+m}{n} + \sum_{j=1}^{m-1} u_n v_j \left[\binom{m-1}{j-1}\binom{n+m}{n} + \binom{m-1}{j}\binom{m+n-1}{m}\right] +$$
$$\sum_{i=1}^{n-1} u_i v_0 \binom{n-1}{i-1}\binom{m+n-1}{m} + \sum_{i=1}^{n-1} u_i v_m \left[\binom{n-1}{i-1}\binom{n+m}{n} + \binom{n-1}{i}\binom{n+m-1}{n}\right] +$$
$$\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u_i v_j \left[\binom{n-1}{i-1}\binom{m-1}{j-1}\binom{n+m}{n} + \binom{n-1}{i-1}\binom{m-1}{j}\binom{n+m-1}{m}\right] +$$
$$\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u_i v_j \left[\binom{n-1}{i}\binom{m-1}{j-1}\binom{n+m-1}{n}\right]\Bigg\} +$$
$$Y(n+m)\Bigg\{ u_0 v_0 \binom{n+m}{n} + u_0 v_m \binom{n+m-1}{m} +$$
$$\sum_{j=1}^{m-1} u_0 v_j \left[\binom{m-1}{j-1}\binom{m+n-1}{m} + \binom{m-1}{j}\binom{n+m}{n}\right] +$$
$$u_n v_0 \binom{m+n-1}{n} + \sum_{j=1}^{m-1} u_n v_j \binom{m-1}{j}\binom{m+n-1}{n} +$$
$$\sum_{i=1}^{n-1} u_i v_0 \left[\binom{n-1}{i-1}\binom{n+m-1}{n} + \binom{n-1}{i}\binom{n+m}{n}\right]$$
$$+\sum_{i=1}^{n-1} u_i v_m \binom{n-1}{i}\binom{n+m-1}{m} + \sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u_i v_j \binom{n-1}{i-1}\binom{m-1}{j}\binom{n+m-1}{n} +$$
$$\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} u_i v_j \left[\binom{n-1}{i}\binom{m-1}{j-1}\binom{n+m-1}{m} + \binom{n-1}{i}\binom{m-1}{j}\binom{n+m}{n}\right]\Bigg\}.$$

Now expanding,

$$\begin{aligned}
U\cdot V &= u_0v_0\binom{n+m}{n}x_1^{[n+m]}+u_0v_mx_0^{[m]}x_1^{[n]}+\sum_{j=1}^{m-1}u_0v_j\binom{n+m-j}{n}x_0^{[j]}x_1^{[n+m-j]}\\
&+u_nv_0x_0^{[n]}x_1^{[m]}+u_nv_m\binom{n+m}{n}x_0^{[n+m]}+\sum_{j=1}^{m-1}u_nv_j\binom{n+j}{n}x_0^{[n+j]}x_1^{[m-j]}\\
&+\sum_{i=1}^{n-1}u_iv_0\binom{n+m-i}{m}x_0^{[i]}x_1^{[n+m-i]}+\sum_{i=1}^{n-1}u_iv_m\binom{m+i}{i}x_0^{[m+i]}x_1^{[n-i]}\\
&+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}u_iv_j\binom{i+j}{i}\binom{(n+m)-(i+j)}{n-i}x_0^{[i+j]}x_1^{[(n+m)-(i+j)]}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\psi(U\cdot V) = X(n+m)\Bigg\{ & u_0v_m\binom{n+m-1}{n}+\sum_{j=1}^{m-1}u_0v_j\binom{n+m-j}{n}\binom{n+m-1}{j-1}+\\
& u_nv_0\binom{n+m-1}{m}+u_nv_m\binom{n+m}{n}+\sum_{i=1}^{n-1}u_iv_m\binom{m+i}{i}\binom{m+n-1}{n-i}+\\
& \sum_{j=1}^{m-1}u_nv_j\binom{n+j}{n}\binom{n+m-1}{m-j}+\sum_{i=1}^{n-1}u_iv_0\binom{n+m-i}{m}\binom{n+m-1}{i-1}\\
& +\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}u_iv_j\binom{i+j}{i}\binom{(n+m)-(i+j)}{n-i}\binom{n+m-1}{i+j-1}\Bigg\}\\
+Y(n+m)\Bigg\{ & u_0v_0\binom{n+m}{n}+u_0v_m\binom{n+m-1}{m}+\sum_{j=1}^{m-1}u_0v_j\binom{n+m-j}{n}\binom{n+m-1}{j}+\\
& u_nv_0\binom{n+m-1}{n}+\sum_{j=1}^{m-1}u_nv_j\binom{n+j}{n}\binom{n+m-1}{n+j}+\sum_{i=1}^{n-1}u_iv_0\binom{n+m-i}{m}\binom{n+m-1}{i}+\\
& \sum_{i=1}^{n-1}u_iv_m\binom{i+m}{i}\binom{m+n-1}{m+i}+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}u_iv_j\binom{i+j}{i}\binom{(n+m)-(i+j)}{n-i}\binom{n+m-1}{i+j}\Bigg\}.
\end{aligned}$$

We show that $\psi(U\cdot V)=\psi(u)\cdot\psi(V)$ for all polynomials U,V. In order to verify this equation, we need the equality of the coefficients of u_iv_j in the both sides of this equation. We see that the coefficients of $u_iv_j, i=0,\dots,n$ and $j=0,\dots,n$ in the both sides of the equation are equal for $X(n+m)$ as well as $Y(n+m)$. Then ψ is a $\mathbb{Z}$-algebra homomorphism.

We will show that the $\mathbb{Z}$-algebra homomorphism ψ is surjective. Because, for every element $aX(n)+bY(n)\in H^{2n}(LSU_2/T,\mathbb{Z})$, we have $a\,x_0^{[n]}+b\,x_1^{[n]}$ such that $\psi(a\,x_0^{[n]}+b\,x_1^{[n]})=aX(n)+bY(n)$, where $a,b\in\mathbb{Z}$.

Now we want to find the kernel of the homomorphism ψ. For $n,m\geq 1$, let

$$u_{n,m}=x_0^{[n]}\cdot x_1^{[m]}-\binom{n+m-1}{m}x_0^{[n+m]}-\binom{n+m-1}{n}x_1^{[n+m]}. \tag{4.39}$$

We claim that the kernel of the homomorphism ψ is equal to the following ideal $I_{\mathbb{Z}}$ generated by the elements $u_{n,m}$.

$$I_{\mathbb{Z}}=\sum_{k\geq 2}I_{\mathbb{Z}}^k,$$

where

$$I_{\mathbb{Z}}^k=\left\{\sum_{0<r<k}t_r^k\left(x_0^{[r]}x_1^{[k-r]}-\binom{k-1}{k-r}x_0^{[k]}-\binom{k-1}{r}x_1^{[k]}\right):t_r^k\in\Gamma_{\mathbb{Z}}(x_0,x_1)\right\}.$$

Now we will prove that our claim is true. Let $U\in I_{\mathbb{Z}}^k$. Then

$$\begin{aligned}\psi(U) &= \psi\left(\sum_{0<r<k}t_r^k(x_0^{[r]}x_1^{[k-r]}-\binom{k-1}{k-r}x_0^{[k]}-\binom{k-1}{r}x_1^{[k]})\right)\\ &= \sum_{0<r<k}\psi(t_r^k)\cdot\psi\left(x_0^{[r]}x_1^{[k-r]}-\binom{k-1}{k-r}x_0^{[k]}-\binom{k-1}{r}x_1^{[k]}\right).\end{aligned}$$

Then $\psi(U)$ is equal to

$$\sum_{0<r<k}\psi(t_r^k)\left(\left(\binom{k-1}{k-r}X(k)+\binom{k-1}{r}Y(k)-\binom{k-1}{k-r}X(k)-\binom{k-1}{r}Y(k)\right)\right.$$

Then $\psi(U)=0$. So, $U\in\ker\psi$.

Conversely, let $U = \sum_{i=0}^{k} u_i x_0^{[i]} x_1^{[k-i]} \in \ker\psi$. Then,

$$\psi(U) = u_0 Y(k) + u_k X(k) + \sum_{i=1}^{k-1} u_i \left[\binom{k-1}{k-i} X(k) + \binom{k-1}{i} Y(k) \right] = 0,$$

So, we have to determine the solution of the homogeneous linear equations system $A \cdot v = 0$, where

$$A = \begin{pmatrix} 1 & k-1 & \ldots & \binom{k-1}{i} & \ldots & 1 & 0 \\ 0 & 1 & \ldots & \binom{k-1}{k-i} & \ldots & k-1 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ u_{k-1} \\ u_k \end{pmatrix}.$$

The rank of the matrix A is 2, so we have infinite solution vectors which have $k-1$ linear independent components and other two components depend these linear independent components. Then,

$$v = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ u_{k-1} \\ u_k \end{pmatrix} = \begin{pmatrix} -\sum_{i=1}^{k-1} t_i \binom{k-1}{i} \\ t_1 \\ \vdots \\ t_i \\ \vdots \\ t_{k-1} \\ -\sum_{i=1}^{k-1} t_i \binom{k-1}{k-i} \end{pmatrix},$$

where $t_i \in \mathbb{Z}$ for $i = 1, \ldots, k-1$. So, $U \in \ker\psi$ is given by

$$\begin{aligned} U &= -\sum_{i=1}^{k-1} t_i \binom{k-1}{i} x_1^{[k]} - \sum_{i=1}^{k-1} t_i \binom{k-1}{k-i} x_0^{[k]} + \sum_{i=1}^{k-1} t_i x_0^{[i]} x_1^{[k-i]} \\ &= \sum_{i=1}^{k-1} t_i \left(x_0^{[i]} x_1^{[k-i]} - \binom{k-1}{k-i} x_0^{[k]} - \binom{k-1}{i} x_1^{[k]} \right) \end{aligned}$$

for some $t_i \in \mathbb{Z}$. Thus, we have proved that $U \in I_{\mathbb{Z}}^k$. □

Theorem 4.6. *Under the isomorphism ψ, the $\mathbb{Z}$-module BGG-operator A^i of $H^*(LSU_2/T, \mathbb{Z})$ corresponds to the partial derivation operator*

$$\begin{cases} \frac{\partial}{\partial x_j} & \text{for degree } 4n \\ \frac{\partial}{\partial x_i} & \text{for degree } 4n+2 \end{cases}$$

for $i \neq j$, $i = 0, 1$.

Proof. We will prove that $\mathbb{Z}$-cohomology operator A^i corresponds to the partial derivation operators as stated. By definition of A^i, we have

$$\begin{aligned}
A^0 \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n} &= 0, \\
A^1 \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n} &= \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^{n-1} r_{\mathbf{a}_0}}, \\
A^0 \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n r_{\mathbf{a}_0}} &= \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n}, \\
A^1 \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n r_{\mathbf{a}_0}} &= 0, \\
A^0 \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n} &= \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^{n-1} r_{\mathbf{a}_1}}, \\
A^1 \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n} &= 0, \\
A^0 \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}} &= 0, \\
A^1 \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}} &= \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n}.
\end{aligned}$$

By ψ isomorphism, we have the following correspondences:

$$\begin{aligned}
\varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n} &\longleftrightarrow x_0^{[2n]}, & \varepsilon^{(r_{\mathbf{a}_0} r_{\mathbf{a}_1})^n r_{\mathbf{a}_0}} &\longleftrightarrow x_0^{[2n+1]}, \\
\varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n} &\longleftrightarrow x_1^{[2n]}, & \varepsilon^{(r_{\mathbf{a}_1} r_{\mathbf{a}_0})^n r_{\mathbf{a}_1}} &\longleftrightarrow x_1^{[2n+1]}.
\end{aligned}$$

The last equations and correspondences verify our claim. □

Corollary 4.7. *The partial derivation operator* $\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}$ *on the divided power algebra induces a derivation on cohomology of* LSU_2/T*.*

Now we will discuss cohomology of ΩG respect to LG/T and G/T where G is a compact semi-simple Lie group. Since ΩG is homotopic to Ω_{pol}, the discussion can be restricted to the Kač-Moody groups and homogeneous spaces. The Lie algebras of $L_{\text{pol}}G_{\mathbb{C}}/B^+$, $L_{\text{pol}}G_{\mathbb{C}}/G_{\mathbb{C}}$ and $G_{\mathbb{C}}/B$ are $\mathbf{g}[\mathbf{t},\mathbf{t^{-1}}]/\mathbf{b}^+$, $\mathbf{g}[\mathbf{t},\mathbf{t^{-1}}]/\mathbf{g}$ and $\mathbf{g}/\mathbf{b}$ respectively. There is a surjective homomorphism

$$\text{ev}_{t=1} : \mathbf{g}[\mathbf{t},\mathbf{t^{-1}}]/\mathbf{b}^+ \to \mathbf{g}/\mathbf{b},$$

with $\ker \text{ev}_{t=1} = \mathbf{g}[\mathbf{t},\mathbf{t^{-1}}]/\mathbf{g}$. Since the odd cohomology groups of $\mathbf{g}[\mathbf{t},\mathbf{t^{-1}}]/\mathbf{b}^+$ and $\mathbf{g}/\mathbf{b}$ are trivial, the second term E_2^{**} of the Leray-Serre spectral sequence collapses and hence we have

Theorem 4.8. *Let* R *is a commutative ring with unit. Then there exists an injective homomorphism* $j : H^*(G/T,R) \to H^*(LG/T,R)$ *and a surjective homomorphism* $i : H^*(LG/T,R) \to H^*(\Omega G,R)$*. In particular,* $J = imj^+$ *is an ideal of* $H^*(LG/T,R)$ *and*

$$H^*(\Omega G,R) \cong H^*(LG/T,R)//J.$$

Theorem 4.9.

$$H^*(\Omega SU_2,R) \cong \Gamma_R(x,y)/\Big(I_R, a(x^{[1]} - y^{[1]})\Big) \cong \Gamma_R(x),$$

where $a \in R$*.*

Now we will give a different approach to determine the cohomology ring of based loop group ΩG using the Schubert calculus. For a compact simply-connected semi-simple Lie group G, we have from [13].

Theorem 4.10. *The natural map*

$$G \to LG \to LG/G \cong \Omega G,$$

is a split extension of Lie groups.

Theorem 4.11. *Let G be a compact simply-connected semi-simple Lie group and let T be a maximal torus of G. Then $\pi : LG/T \to LG/G$ is a fiber bundle with the fibre G/T.*

Proof. Since $LG \to LG/G$ is a principal G-bundle and G/T is a left G-space by the action $g_1 \cdot g_2T = g_1g_2T$ for $g_1, g_2 \in G$, we have a fibration

$$G/T \to LG \times_G G/T \to \Omega G.$$

Therefore, we have to show that $LG \times_G G/T$ is diffeomorphic to LG/T. Since $LG \times_G G/T$ is equal to

$$\{[\gamma, gT] : [\gamma, gT] = [\gamma h, h^{-1}gT]\, \forall g, h \in G, \gamma \in LG\},$$

we define a smooth map $\tau : LG \times_G G/T \to LG/T$ given by $[\gamma, gT] \to \gamma gT$. It is well-defined because for $h \in G$,

$$\begin{aligned} \tau([\gamma h, h^{-1}gT]) &= \gamma hh^{-1}gT \\ &= \gamma gT \\ &= \tau([\gamma, gT]). \end{aligned}$$

For every γT, we can find an element $[\gamma, T] \in LG \times_G G/T$ such that $\tau([\gamma, T]) = \gamma T$. So, τ is a surjective map. Now, we will show that τ is an injective map. Let $[\gamma_1, g_1T], [\gamma_2, g_2T] \in LG \times_G G/T$ such that

$$\tau([\gamma_1, g_1T]) = \tau([\gamma_2, g_2T]). \tag{4.40}$$

The equation Eq.(4.40) gives

$$\gamma_1 g_1 T = \gamma_2 g_2 T.$$

So, $(\gamma_1 g_1)^{-1}(\gamma_2 g_2), (\gamma_2 g_2)^{-1}(\gamma_1 g_1) \in T$. Then,

$$[\gamma_1, g_1T] \quad = \quad [\gamma_1 g_1, g_1^{-1}g_1T]$$

$$
\begin{aligned}
&= [\gamma_1 g_1, T] \\
&= [(\gamma_1 g_1)(\gamma_1 g_1)^{-1}(\gamma_2 g_2), (\gamma_2 g_2)^{-1}(\gamma_1 g_1)T] \\
&= [\gamma_2 g_2, T] \\
&= [\gamma_2 g_2 g_2^{-1}, g_2 T] \\
&= [\gamma_2, g_2 T].
\end{aligned}
$$

Thus, we proved that τ is an injective map and it's inverse is given by $\gamma T \to [\gamma, T]$ which is smooth map. Then, $\pi : LG/T \to LG/G = \Omega G$ given by $\gamma T \to \gamma G$ is a fiber bundle map. □

Since LG/T is a fiber bundle over ΩG with the fiber G/T, by the Leray-Serre spectral sequence of the fibration and Corollary (5.13) of Kostant and Kumar [9], $\theta : H^*(\Omega G, \mathbb{Z}) \to H^*(LG/T, \mathbb{Z})$ is injective and $\theta(H^*(\Omega G, \mathbb{Z}))$ is generated by the Schubert classes $\{\varepsilon^w\}_{w \in \widehat{W}}$ in the cohomology of LG/T and hence we can determine the cohomology ring of ΩG.

Let R be a commutative ring with unit and let $\Gamma_R(\gamma)$ be the divided power algebra with $\deg \gamma = 2$.

Theorem 4.12. *$H^*(\Omega SU(2), R)$ is isomorphic to $\Gamma_R(\gamma)$ with the R-module basis $\gamma^{[n]}$ in each degree $2n$ for $n \geq 1$.*

Proof. Since the odd cohomology is trivial, by the universal coefficient theorem, it suffices to prove this for $R = \mathbb{Z}$. The integral cohomology of ΩSU_2 is generated by the Schubert classes indexed

$$\widehat{W} = \{\overline{\ell(w)} : w \in \widetilde{W}\} = \{(r_{\mathbf{a_0}} r_{\mathbf{a_1}})^n, \, (r_{\mathbf{a_0}} r_{\mathbf{a_1}})^n r_{\mathbf{a_0}} : n \geq 0\}.$$

Then, we define a $\mathbb{Z}$-algebra homomorphism η from $\Gamma_{\mathbb{Z}}(\gamma)$ to $H^*(\Omega SU_2, \mathbb{Z})$ given as follows. For $n \geq 0, u_n \in \mathbb{Z}$, $\eta(u_n \gamma^{[n]}) = u_n X(n)$. Now, we will show that η is a $\mathbb{Z}$-algebra homomorphism. We have

$$\eta\left(\gamma^{[n]} \cdot \gamma^{[m]}\right) = \eta\left(\binom{n+m}{n} \gamma^{[n+m]}\right)$$

$$= \binom{n+m}{n} X(n+m).$$

Let us calculate $\eta(\gamma^{[n]}) \cdot \eta(\gamma^{[m]}) = X(n) \cdot X(m)$. By equations Eq.(4.25) and Eq.(4.27), we have

$$X(n) \cdot X(m) = \binom{n+m}{n} X(n+m).$$

So,

$$\eta(\gamma^{[n]}) \cdot \eta(\gamma^{[m]}) = \binom{n+m}{n} X(m+n).$$

Then, we have shown that η is a $\mathbb{Z}$-algebra homomorphism.

Also, it is surjective and injective. Because, for every element $u_n X(n) \in H^*(\Omega SU_2, \mathbb{Z})$, we have $u_n \gamma^n$ such that $\eta(u_n \gamma^n) = u_n X(n)$ and

$$\begin{aligned} \ker \eta &= \{u_n \gamma^n : \eta(u_n \gamma^n) = u_n X(n) = 0\} \\ &= \{u_n \gamma^n : u_n = 0\} \\ &= 0. \end{aligned}$$

We have completed the proof. $\square$

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